ANALYSIS AND COMPUTATION OF MULTIPLE UNSTABLE SOLUTIONS TO NONLINEAR ELLIPTIC SYSTEMS

A Dissertation

by

XIANJIN CHEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2008

Major Subject: Mathematics

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ABSTRACT

Analysis and Computation of Multiple Unstable Solutions to
Nonlinear Elliptic Systems. (August 2008)

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We study computational theory and methods for finding multiple unstable solutions (corresponding to saddle points) to three types of nonlinear variational elliptic systems: cooperative, noncooperative, and Hamiltonian. We first propose a new L-orthogonal selection in a product Hilbert space so that a solution manifold can be defined. Then, we establish, respectively, a local characterization for saddle points of finite Morse index and of infinite Morse index. Based on these characterizations, two methods, called the local min-orthogonal method and the local min-max-orthogonal method, are developed and applied to solve those three types of elliptic systems for multiple solutions. Under suitable assumptions, a subsequence convergence result is established for each method. Numerical experiments for different types of model problems are carried out, showing that both methods are very reliable and efficient in computing coexisting saddle points or saddle points of infinite Morse index. We also analyze the instability of saddle points in both single and product Hilbert spaces. In particular, we establish several estimates of the Morse index of both coexisting and non-coexisting saddle points via the local min-orthogonal method developed and propose a local instability index to measure the local instability of both degenerate and nondegenerate saddle points. Finally, we suggest two extensions of an L-orthogonal selection for future research so that multiple solutions to more general elliptic systems such as nonvariational elliptic systems may also be found in a stable way.

To the memory of my father

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CHAPTER I

INTRODUCTION

This dissertation is to study the computational theory and methods for three types of nonlinear elliptic systems: cooperative, noncooperative, and Hamiltonian. A framework for characterizing and computing multiple unstable solutions is developed.

A. Motivation

An understanding of the interaction of simple physical objects (e.g., scalar solitons) leading to the formation of more complex objects (e.g., vector solitons) is an ultimate goal [37] of the fundamental research in many key physics areas such as condensed matter physics, dynamics of biomolecules, nonlinear optics, etc. Mathematically, those complex objects may be described by certain differential systems which give rise to nonlinear elliptic systems.

Exhibiting many novel new phenomena that are not present in single equation case, systems are much more interesting in many applications. Systems that are Euler-Lagrange equations of some functionals are called variational and can be treated using the Critical Point Theory, since their (weak) solutions are critical points of the functionals that originate them. Solutions corresponding neither to local maxima nor to local minima are the so-called saddle points. They appear as unstable equilibria or transient excited states in many physical systems and are usually thought to be too difficult to capture. Due to their diversity in nature, variational systems can be classified in many different ways. In this thesis, we deal with three types of them (see also Section I.B.2): cooperative, noncooperative, and Hamiltonian.

This dissertation follows the style of *Mathematics of Computation*.

Variational methods especially the minimax methods [49,57,61] have been among the main methods in obtaining the existence or multiplicity of critical (saddle) points. In particular, over the last several decades, the minimax methods (theorems) have been often used to locate saddle points by means of appropriate min-max procedures, provided that the Palais-Smale condition and other suitable geometric structures of the functionals are met. However, most minimax theorems in the literature such as the Mountain Pass Theorem, the Saddle Point Theorem, and Linking Theorems characterized a saddle point as a solution to a two-level global optimization (minimax) problem which is a very expensive or even impossible job from a numerical point of view. Therefore, those minimax theorems cannot be directly applied to find saddle points. To circumvent this difficulty, there is a need to establish some alternative characterizations on saddle points based on which numerical methods or algorithms can be feasibly designed and implemented. Meanwhile, due to the diversity and complexity of systems, there is no general method or algorithm available for solving them all. This means that we more or less need to establish a characterization as well as a numerical method for each class individually.

Highly or multiply excited states which can now be electronically excited or laser-induced have been observed or demonstrated [15,32,43,51,58] in many physics areas including quantum mechanics, condensed matter physics, nonlinear optics, etc. Those excited states can easily decay into lower-lying excited states or the ground states (i.e., the minimum energy states) under small perturbations. Meanwhile, those highly excited states enjoy a large variety of configurations and maneuverabilities, which may lead to many promising or even surprising applications. With new atomic, optical or synchrotronic technologies, scientists now are able to successfully trap or secure them and search for their potential applications [13,34,53,54,58]. These new technologies and advanced studies are changing our traditional views of unstable solutions and starting to draw more and more attention from both scientists and engineers.

For example, spatial solitons have been a subject of many studies [26,27,35,37,41] since their first theoretical prediction [18]. After a number of experimental observations of self-guided light beams in various types of nonlinear bulk media were reported, the study of spatial optical solitons and their interactions became an active research area in nonlinear optics, see [26,27,32,35,37]. In particular, it has been shown that several light beams can be combined to produce multicomponent self-trapped states, so-called spatial vector solitons. Physically, these vector solitons (corresponding to co-existing excited states) are "particle-like" localized nonlinear objects; mathematically, they are unstable standing solitary wave solutions to certain nonlinear Schrödinger (NLS) systems [26,27,32], see also (1.4). Since all those vector solitons are unstable, instability analysis becomes important both practically and theoretically. However, "so far, numerical methods have been proved to be the only available tool for analyzing the mutually trapped states in the nonlinear regime, especially solitons without radial symmetry (e.g., dipole or multipole vector solitons)"[27]. It has also been observed that the dipole-mode vector solitons are much more stable than any other vector soliton. They are "stable enough for experimental observation, ..., extremely robust, have a typical lifetime of several hundred diffraction lengths and survive a wide range of perturbations"[32]. As a result, there is a growing need to develop more efficient and reliable numerical methods for finding those multiple (coexisting) excited states, since analytic solution expressions are generally too difficult to obtain.

Numerically, motivated by the Mountain Pass Theorem, the first ingenious algorithm (also called mountain pass algorithm) devoted to computing saddle points (actually the ground states) was proposed by Choi-McKenna [19] in 1993. Then a high linking method was developed by Ding-Costa-Chen [28] in 1999. But no formal mathematical justification or convergence on those methods was provided. After that, a local minimax method (LMM), also the first method which can capture multiple saddle points in a certain order, was developed by Li-Zhou [38,39] in 2001. Mathematical justification and convergence on LMM were also established therein. In general, however, these three methods can solve nonlinear partial differential equations (PDEs) in single equation case rather than in system case. Take LMM for example. When used to solve systems, it gives no help for distinguishing coexisting solutions from non-coexisting ones. As a result, one's efforts in locating those coexisting also desired solutions will be greatly weakened. Moreover, for noncooperative and Hamiltonian elliptic systems, so far there is no method available to solve them in the literature due to the strongly indefinite (see Definition I.2) nature of the functionals that originate them. Nevertheless, one may want to mention the Newton's method. But it is known that the Newton's method depends heavily on initial guesses and has difficulties in handling degeneracy which arises quite naturally in multiple solution problems. Besides, even it successfully captures a solution, it provides no instability information for the solution since it does not assume or use the variational structure of those systems. Therefore, new numerical methods and strategies need to be developed.

Our goal is to develop some reliable and efficient numerical methods for variational elliptic systems for multiple solutions, to introduce new numerical techniques in nonlinear PDEs, and to analyze the stability and instability of multiple solutions captured, which are both inspired by and shall shed some new light upon the critical point theory and variational methods as well.

In the rest of this chapter, we recall some fundamental results on abstract critical point theorems as well as three aforementioned numerical methods for saddle point search. We also give a classification of nonlinear variational elliptic systems. In Chapter II, we present a local min-orthogonal method for cooperative elliptic systems.

After proposing a new L-$\perp$ selection as well as a new stable solution manifold $\mathcal{M}$, we establish a local min-orthogonal characterization for coexisting saddle points of a positive definite functional and show that a local minimum of the functional on such manifold $\mathcal{M}$ corresponds to a saddle point of the functional in the entire space. We then design a local min-orthogonal algorithm and give a subsequence convergence result for the algorithm. A numerical technique on computing the gradients is also introduced in this chapter.

In Chapter III, we develop a local min-max-orthogonal method for noncooperative elliptic systems. We start from the observation that although functionals associated to noncooperative elliptic systems are strongly indefinite, they are positive definite in one variable and are negative definite in another. Taking advantage of such important feature of the functionals, we establish a local min-max-orthogonal characterization for saddle points of infinite Morse index based on which a local min-max-orthogonal algorithm together with its subsequence convergence is developed. In addition, we show that a game-type saddle point of a strongly indefinite functional J on an induced solution manifold $\mathcal{M}$ is also a saddle point of J in the entire space H. Consequently, instead of searching for infinite Morse index saddle points of J in H, we can just keep our eyes on those saddle points of J on $\mathcal{M}$.

In Chapter IV, we carry out some instability analysis on saddle points. We extend some instability analysis results in [67] via a local min-orthogonal method in two directions: from a local peak selection to a local L-$\perp$ selection and from a single Hilbert space (corresponding to a single equation case) to a product Hilbert space (corresponding to a system case). Based on a local min-orthogonal characterization, estimates on the Morse index are established for saddle points in both single and product Hilbert spaces. In addition, a local instability index for saddle points in a product Hilbert space is discussed and used to induce an order for multiple unstable solutions (saddle points) captured.

In Chapter V, we apply the local min-orthogonal method and the local min-max-orthogonal method developed in Chapters II-III and the instability analysis results established in Chapter IV to solve three types of elliptic systems for multiple solutions and give numerical estimates of the Morse index as well. We carry out numerical experiments for various model problems including the 2- and 3-component vector soliton problems, two noncooperative elliptic systems, and two Hamiltonian elliptic systems (i.e., the Lane-Emden system and the nonlinear biharmonic problem). We visualize our numerical solutions to those problems via contour plots and/or profiles and verify several important properties (e.g., existence, differentiability, separation) of an L-$\perp$ selection and a solution manifold $\mathcal{M}$ induced. Moreover, we show that saddle points of certain strongly indefinite functional are still saddle points even when the functional is confined to the solution manifold $\mathcal{M}$. This implies that saddle points of this kind can be approximated by the local min-max-orthogonal method but not by the local min-orthogonal method. For a special class of Hamiltonian elliptic systems, we state their close relationship with noncooperative elliptic systems.

Finally, we conclude in Chapter VI with a brief description of how the preceding results may be further extended to solve more general elliptic systems such as nonvariational systems. We indicate that the L-$\perp$ selection plays a significant role in both characterizations and computations of multiple solutions and can be extended in two directions so that more multiple solution problems can be attacked in the future.

B. Preliminaries

In this section we start with global characterizations on saddle points by recalling some abstract critical point theorems. We then give a classification of nonlinear variational elliptic systems according to the nature of their associated functionals and list some model problems as well. Finally, three numerical methods (algorithms) for nonlinear PDEs are recalled.

1. Abstract Critical Point Theorems

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, $I : H \to \mathbb{R}$ be a C^1-functional (i.e., continuously differentiable) and I' be its first Frechet derivative. A point $u^* \in H$ is called a critical point of I if $I'(u^*) = 0$, i.e.,

$$I'(u^*)\phi = 0$$

for all $\phi \in H$. In this case, $c := I(u^*)$ is called a critical value of I at u^*. A critical point u^* of I is called nondegenerate if the second Frechet derivative of I at u^* or $I''(u^*)$ exists and has a bounded inverse; otherwise, u^* is called degenerate.

For a critical point u^* of I, assume $I''(u^*)$ is a self-adjoint Fredholm operator from $H \to H$. According to the spectral theory, H has an orthogonal spectral decomposition

$$H = H^- \oplus H^0 \oplus H^+$$

where H^-, H^0, H^+ are respectively the maximum negative definite, the null and the maximum positive definite subspaces of $I''(u^*)$ in H with $\dim(H^0) < \infty$, and are invariant under $I''(u^*)$.

Definition I.1 ([8]) *The Morse index (MI) of a critical point u^* of I is the dimension of the maximum negative definite subspace of $I''(u^*)$, i.e., MI$(u^*) = \dim(H^-)$.*

Evidently, a nondegenerate critical point u^* with $\mathrm{MI}(u^*) = 0$ is a local minimum of I and corresponds generally to a stable solution; while a critical point u^* of I with $\mathrm{MI}(u^*) > 0$ represents an unstable solution.

Definition I.2 ([2],[1],[7]) *Let $\phi \in C^1(H, \mathbb{R})$ be a functional of the form*

$$\phi(u) = \frac{1}{2}\langle Au, u\rangle + b(u)$$

where $A : H \to H$ is a self-adjoint invertible linear operator and b is a functional (usually nonlinear) with compact gradient $\nabla b \in C(H, H)$ (i.e., it maps bounded sets into relatively compact sets). Such functional ϕ is called strongly indefinite if both the positive and negative eigenspaces of A are infinite-dimensional; if the dimension of the negative eigenspace is zero (finite), then it is called positive (semi-positive) definite.

Obviously, if a functional ϕ is strongly indefinite, so is $-\phi$. In this case, every critical point of both ϕ and $-\phi$ has infinite Morse index. Hence the classical Morse theory (see, e.g., [8]) is no longer applicable, interested readers may refer to [2,1]. Definition I.2 also implies that a strongly indefinite functional is neither bounded from above nor from below, not even modulo a finite-dimensional subspace [49]. Moreover, if the Hessian $\phi''(x)$ of ϕ at some $x \in H$ exists, then it is a Fredholm operator [2].

The formulation of abstract critical point theorems generally requires some compactness hypotheses for a functional I. A condition of that type that has proved to be very useful in obtaining the existence or multiplicity of critical points was first introduced by Palais and Smale (see [49] and references therein) in their work on Morse theory in infinite-dimensional spaces. Let X be a Banach space and $I : X \to \mathbb{R}$ a C^1-functional.

Definition I.3 *A functional I is said to satisfy the Palais-Smale condition, denoted by (PS), if any sequence $\{u_m\}$ in X such that $I(u_m)$ is bounded and $I'(u_m) \to 0$ (in the strong sense) admits a convergent subsequence.*

The minimax theorems [49,57,61] usually characterize a critical value c of I as a minimax:

$$c := \inf_{A \in \mathcal{A}} \max_{u \in A} I(u) \ (global) \tag{1.1}$$

where $\mathcal{A}$ is some class of compact subsets of X chosen to take advantage of the topological structure of the functional I. Very often the minimax has a related form

$$c := \inf_{\gamma \in \Gamma} \max_{\theta \in M} I(\gamma(\theta)) \quad \text{or} \quad c := \inf_{A=\gamma(M) \in \mathcal{A}} \max_{u \in A} I(u) \ (global)$$

where M is chosen to be a compact subset in some parameter space, $\gamma \in C(M, X)$, i.e., Γ is a family of continuous maps from M to X satisfying appropriate properties. Earliest results (e.g., the Ljusternik-Schnirelmann theory [49,57]) of this type mainly studied functionals which are bounded from below, and were invariant under some group of symmetries, and were defined on a Banach manifold which was invariant under those symmetries, see a survey paper [49] by P.H. Rabinowitz. In 1970s, the minimax theorems were extended by Ambrosetti and Rabinowitz to functionals which are not necessarily bounded from below, e.g., functionals which are strongly indefinite.

As one of the simplest and most geometrically appealing minimax theorems in the literature, the Mountain Pass Theorem proved by Ambrosetti and Rabinowitz [3] in 1973 states that

Theorem I.1 (Mountain Pass Theorem) *Let X be a real Banach space and $\phi \in C^1(X, \mathbb{R})$ satisfying the (PS) condition with $\phi(0) = 0$. Assume that*

(1) $\exists \rho, \alpha > 0$ s.t. $\phi(u) \geq \alpha, \forall u$ with $\|u\| = \rho$ and

(2) $\exists e \in X$ with $\|e\| > \rho$ s.t. $\phi(e) \leq \phi(0) = 0$.

Then

$$c = \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} \phi(\gamma(\theta)) \geq \alpha$$

is a critical value of ϕ, where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$.

The mountain pass theorem set undoubtedly a milestone in both modern nonlinear analysis and critical point theory. It influenced its "successors" so much that it can be considered as the beginning of a new era in critical point theory. The theorem was so insightful that it was often used as a model or framework for a number of other critical point theorems such as the Saddle Point Theorem [48] and Linking Theorems (see [9,49] and references therein) that followed. The following is one version of linking theorems due to Rabinowitz.

Theorem I.2 (Linking Theorem) *Let B be a Banach space with splitting $B = L \oplus X$, where X, L are two closed subspaces of B with $\dim(L) < \infty$. Assume that $I \in C^1(B, \mathbb{R})$ satisfies the (PS) condition and*

(1) *there are $\rho, \alpha > 0$ such that $I(v) \geq \alpha$, $\forall v \in \partial B_\rho \cap X$,*

(2) *there are $u \in X$ with $\|u\| = 1$ and a number $R > \rho$ such that $I(v) \leq 0$, $\forall v \in \partial Q$, where $Q = (\bar{B}_R \cap L) \oplus \{ru | r \in (0, R)\}$.*

Then

$$c := \inf_{\Gamma} \max_{v \in Q} I(h(u))$$

is a critical value of I, where $\Gamma = \{h \in C(\bar{Q}, B) | h = id \text{ on } \partial Q\}$.

Note that in other versions of linking theorems (see, e.g., [9]), both subspaces L, X of B may be infinite-dimensional. In this case, the functional I becomes strongly indefinite.

2. A Classification on Elliptic Systems

In this section, we classify nonlinear variational elliptic systems according to the nature of their associated functionals and state some model problems as well. Note that due to the diversity and complexity of systems, we do not intend to classify and include all of them.

It is known that nonlinear elliptic systems arise quite extensively in describing self-organized natural phenomena, ranging from simple physical systems to complex biological processes. Among them, there are at least four types of elliptic systems that can be treated variationally: cooperative, noncooperative, Hamiltonian type I and Hamiltonian type II. We state these four types of elliptic systems as below.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $G : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be of class C^1 in the second and third variables, $\frac{\partial}{\partial n}$ denote the outer normal derivative.

(I) Cooperative Elliptic Systems

Cooperative elliptic systems can be expressed as

$$\begin{cases} -\Delta u = G_u(x,u,v), & x \in \Omega \\ -\Delta v = G_v(x,u,v), & x \in \Omega \\ u = v = 0 \ \text{ or } \ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

where (G_u, G_v) stands for the gradient of G in the variables $(u,v) \in \mathbb{R}^2$. Under some standard growth conditions on $G(x,u,v)$, weak solutions of (1.2) are critical points of the C^1-functional $J : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ or $H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$J(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \int_\Omega G(x,u,v)dx. \tag{1.3}$$

Clearly, the quadratic part of the functional J above is positive definite which, with some suitable assumptions on G, implies that J is also positive definite. The existence of (multiple) nontrivial solutions to systems of the form (1.2) has been established by many authors, see, e.g., [6,46,64,65,68].

To give a specific example of (1.2), let us consider 2-coupled NLS equations

[27,32,45]

$$\begin{cases} i\dfrac{\partial E_1}{\partial z} + \Delta E_1 - \dfrac{E_1}{1+(|E_1|^2+|E_2|^2)} = 0, \\ i\dfrac{\partial E_2}{\partial z} + \Delta E_2 - \dfrac{E_2}{1+(|E_1|^2+|E_2|^2)} = 0, \end{cases} \tag{1.4}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian. Problem (1.4) arises in the study of interaction of two mutually incoherent optical beams (i.e., $E1, E2$) propagating along z-direction in a bulk saturable medium (e.g., photorefractive materials). In study of stability or instability, pattern formation and other vector phenomena, coexisting standing solitary wave solutions (or solitons in short), also called *spatial vector solitons*, to (1.4) of the form

$$E_1 = u(x,y)e^{-i\beta_1 z},\ E_2 = v(x,y)e^{-i\beta_2 z},\ u \neq 0,\ v \neq 0, \tag{1.5}$$

are of particular interest, where $0 < \beta_1 \leq \beta_2 < 1$ are two propagation constants, u, v represent respectively the amplitudes of the beams $E1, E2$. After setting $\lambda_i = 1 - \beta_i$ $(i = 1,2)$, $\gamma = \frac{\lambda_2}{\lambda_1}$ and rescaling the amplitudes, $\{u, v\} \to \{\sqrt{\lambda_1}u, \sqrt{\lambda_1}v\}$, and the coordinates, $\{x, y\} \to \{x/\sqrt{\lambda_1}, y/\sqrt{\lambda_1}\}$, (1.4) leads to a semilinear elliptic system (also one of our model problems to be investigated) [27,32]

$$\begin{cases} -\Delta u &= -u + \frac{uI(u,v)}{1+\mu I(u,v)}, \\ -\Delta v &= -\gamma v + \frac{vI(u,v)}{1+\mu I(u,v)}, \end{cases} \tag{1.6}$$

where $\mu \equiv \lambda_1 = 1 - \beta_1 \in (0,1)$ is the saturation parameter (the case $\mu \to 0$ corresponds to the Kerr medium), $I(u,v) = u^2 + v^2$ is the total intensity, the nonlinear term $\frac{I(u,v)}{1+\mu I(u,v)}$ characterizes a saturable nonlinearity of the medium. In view of (1.2), we see that system (1.6) is variational and

$$G(u,v) = \frac{1-\mu}{2\mu}u^2 + \frac{1-\gamma\mu}{2\mu}v^2 - \frac{\ln(1+\mu I(u,v))}{2\mu^2}. \tag{1.7}$$

Due to the localized nature [27,32] of solitary wave solutions to (1.4), a zero Dirichlet condition on a bounded domain can be imposed for system (1.6). Moreover, one sees that if u or v is zero, then (1.6) reduces to a single equation problem, a situation that needs to be avoided since the coexisting solutions (i.e., coexisting states) to (1.6) are of greater interest to physicists.

Similar coupled NLS equations which also give rise to a system of the form (1.2) appear in other applications. For example, the two circularly polarized optical beams propagating in an isotropic Kerr medium obey the following NLS equations [12]

$$\begin{cases} \frac{\partial U}{\partial z} &= i\frac{1}{2k}\frac{\partial^2 U}{\partial x^2} + i\gamma(|U|^2 + 7|V|^2)U, \\ \frac{\partial V}{\partial z} &= i\frac{1}{2k}\frac{\partial^2 V}{\partial x^2} + i\gamma(|V|^2 + 7|U|^2)V, \end{cases} \tag{1.8}$$

where $U(x,z)$ and $V(x,z)$ are the transverse beam envelopes of the circular polarization components of the electromagnetic field, k is the wave vector modulus in the waveguide and γ is the nonlinear coefficient. As pointed out in [12], equations (1.8) possess a one-parameter family of vector solitons whose amplitude functions also satisfy an elliptic system of the form (1.2).

(II) Noncooperative Elliptic Systems

Noncooperative elliptic systems are of the form

$$\begin{cases} -\Delta u = G_u(x,u,v), & x \in \Omega \\ -\Delta v = -G_v(x,u,v), & x \in \Omega \\ u = v = 0 \ \text{ or } \ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{1.9}$$

where (G_u, G_v) is the gradient of G in the variables $(u,v) \in \mathbb{R}^2$. The associated energy functional is

$$J(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - |\nabla v|^2)dx - \int_\Omega G(x,u,v)dx. \tag{1.10}$$

Multiplicity and existence results for system (1.9) have been established when G

satisfying certain growth conditions, see [20,21,30,47,68]. With Definition I.2, the functional J in (1.10) is strongly indefinite when G has compact gradient.

As an example, consider the reaction-diffusion system (see, e.g., [23,36,44] and references therein)

$$\begin{cases} u_t = D_1\Delta u + f(u) - kv, & x \in \Omega, \quad t > 0 \\ \tau v_t = D_2\Delta v + u - \gamma v, & x \in \Omega, \quad t > 0 \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \quad t \geq 0 \end{cases} \tag{1.11}$$

where u, v are real-valued functions, representing an activator and an inhibitor, respectively; $k, \tau, D_2 > 0$; $\gamma, D_1 = \epsilon^2$ are small positive parameters; $f(u)$ is a cubic function, e.g., $f(u) = u(1-u)(u-a)$, $0 < a < \frac{1}{2}$. System (1.11), also called the FitzHugh-Nagumo type system, was first proposed as a mathematical model of biological pattern formation.

Steady-state (nonconstant) solutions to (1.11) are of particular interest in application and satisfy the following noncooperative elliptic system

$$\begin{cases} D_1\Delta u + f(u) - kv = 0, & x \in \Omega, \\ kD_2\Delta v + ku - k\gamma v = 0, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.12}$$

to which the associated energy functional is

$$J(u,v) = \frac{1}{2}\int_\Omega (D_1|\nabla u|^2 - kD_2|\nabla v|^2)dx + k\int_\Omega uv dx - \frac{k\gamma}{2}\int_\Omega v^2 dx - \int_\Omega F(u)dx. \tag{1.13}$$

Here $F(\cdot)$ is the primitive of $f(\cdot)$.

(III) Elliptic Systems of Hamiltonian Type I

Elliptic systems of Hamiltonian Type I have the form

$$\begin{cases} -\Delta u = G_v(x,u,v), & x \in \Omega, \\ -\Delta v = G_u(x,u,v), & x \in \Omega, \\ u = v = 0 \;\text{ or }\; \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & x \in \partial\Omega. \end{cases} \tag{1.14}$$

With suitable growth conditions on G, weak solutions of (1.14) correspond to critical points of the functional

$$J(u,v) = \int_\Omega \nabla u \nabla v dx - \int_\Omega G(x,u,v)dx \tag{1.15}$$

on a carefully chosen functional space [31,33]. Note that such J is also strongly indefinite and the corresponding functional setting is much more subtle [31,33] (see also Section V.C) in this case than in the two previous ones.

A typical example of system (1.14) is the Lane-Emden system (see, e.g., [10,22]) in which the Hamiltonian G takes the form

$$G(u,v) = u^\alpha + v^\beta, \quad 1 - \frac{2}{N} < \frac{1}{\alpha} + \frac{1}{\beta} \quad \alpha, \beta > 1.$$

Another example of system (1.14) is the nonlinear biharmonic problem

$$\Delta^2 v = |v|^{p-1}v, \quad x \in \Omega \tag{1.16}$$

subject to the Navier boundary conditions $v = \Delta v = 0$ on $\partial\Omega$, which can be rewritten as a system of the form (1.14)

$$\begin{cases} -\Delta u = |v|^{p-1}v, & x \in \Omega, \\ -\Delta v = u, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.17}$$

with $G(u,v)=\frac{1}{2}u^2+\frac{1}{p+1}|v|^{p+1}$.

(IV) Elliptic Systems of Hamiltonian Type II

In contrast with system (1.14), the nonlinear term in this type of elliptic system appears as a source on the boundary rather than in the domain. It reads as [11]

$$\begin{cases} \Delta u = u, \Delta v = v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = G_v(x,u,v), & x \in \partial\Omega, \\ \frac{\partial v}{\partial n} = G_u(x,u,v), & x \in \partial\Omega, \end{cases} \tag{1.18}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $G : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a positive function of class C^1 with suitable growth control on G and its first derivatives. A proper functional setting for (1.18) was established in [11] so that it can be treated variationally.

3. Three Numerical Methods for Nonlinear PDEs

In this section, we recall three saddle point search methods (algorithms) which aim mainly to find unstable solutions to nonlinear PDEs in the single equation case. The first one is proposed by Choi-McKenna [19] in 1993, called the Mountain Pass Algorithm; the second one, called the High Linking Algorithm, is due to Ding-Costa-Chen [28]; the third one, called the Local Minimax Algorithm, is developed by Li-Zhou [38] in 2001. As mentioned before, these three methods (algorithms) have some limitations in search for saddle points in system case, e.g., the coexisting saddle points or saddle points of infinite Morse index. One common feature of these three algorithms is that they can only find saddle points of finite Morse index.

Algorithm I.1 *Mountain Pass Algorithm (Choi-McKenna)*

Step 1. *Take an initial guess* $u_0 \in H_0^1(\Omega)$ *such that* $u_0 \neq 0$ *and* $J(u_0) \leq 0$.

Step 2. *Find* $t^* \in (0,1)$ *such that* $J(t^*u_0) = \max_{t\in[0,1]} J(tu_0)$, *and set* $u_1 = t^*u_0$.

Step 3. *Compute* $\nabla J(u_1)$ *and set* $v = \nabla J(u_1)$.

Step 4. *If* $\|v\| \leq \epsilon$, *then output* u_1 *and stop; else, goto the next step.*

Step 5. *Let* $u = -v + u_1$ *and find* $t^* \in (0,1)$ *such that* $J(t^*u) = \max_{t\in[0,1]} J(tu)$.

Step 6. *If* $J(t^*u) < J(u_1)$, *set* $u_1 = t^*u$ *and goto Step 3; else, set* $v = \frac{1}{2}v$ *and goto Step 5.* ▮

Algorithm I.2 *High Linking Algorithm (Ding-Costa-Chen)*

Step 1. *Find a point* v *such that* $v_0 \neq 0$ *and* $J(v_0) \leq 0$.

Step 2. *Apply the Modified Mountain Pass Method to find a mountain pass solution* v_1 *and* u_1, u_2 *satisfying*

$$J(v_1 + tu_1) < J(v_1), \quad J(v_1 + tu_2) < J(v_1) \quad for\ small\ t \neq 0.$$

Step 3. *Find* $t_1 > 0$ *and* $t_2 < 0$ *such that* $J(v_1 + t_1u_1) \leq 0$ *and* $J(v_1 + t_2u_1) \leq 0$, *and set* $g_1 = v_1 + t_1u_1$ *and* $g_2 = v_1 + t_2u_1$.

Step 4. *Find* $t_3 > 0$ *such that* $J(v_1 + t_3u_2) \leq J(v_1)$, *and set* $g_3 = v_1 + t_3u_2$.

Step 5. *Construct the triangle* $\triangle$ *by*

$$\triangle = \{\lambda_1 g_1 + \lambda_2 g_2 + (1 - \lambda_1 - \lambda_2)g_3 | \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq 1\},$$

and find $v^* \in \triangle$ *such that* $J(v^*) = \max_{g\in\triangle} J(g)$.

Step 6. *If* v^* *is an interior point of* $\triangle$, *then go to next step. Otherwise, set* $u_2 = v^* - v_1$ *and go to Step 4.*

Step 7. *Set* $v_2 = v^*$, *compute* $w = \nabla J(v_2)$.

Step 8. *If* $\|v\| \leq \epsilon$, *then output* v_2 *and stop. Otherwise, set* $u_2 = (-v + v_2) - v_1$ *and go to next step.*

Step 9. *Repeat the same procedures as Step 4-6 to construct a new triangle* $\triangle$ *and find an interior point* $v^* \in \triangle$ *such that* $J(v^*) = \max_{g\in\triangle} J(g)$.

Step 10. *If* $J(v^*) < J(v_2)$, *go to Step 7. Otherwise, set* $w = \frac{1}{2}w$ *and* $u_2 = (-w + v_2) - v_1$, *then go to Step 9.* ▮

Algorithm I.3 *Local Minimax Algorithm (Li-Zhou)*

Assume that $u_1, ..., u_{n-1}$ *are* $n-1$ *critical points (already found) of* J. *Let* $L = [u_1, ..., u_{n-1}]$ *and choose two positive constants* $0 < \lambda < 1$, $\epsilon \ll 1$.

Step 1. *Find an ascent direction* $v_n^1 \in L^\perp$ *at* u_{n-1}.

Step 2. *Solve for*

$$u_n^1 = \sum_{i=1}^{n-1} t_i^1 u_i + t_n^1 v_n^1 = \arg \max_{t_i\in R, i=1,\dots,n-1, t_n>0} J(\sum_{i=1}^{n-1} t_i u_i + t_n v_n^1)$$

with initial point $(0, ..., 0, 1)$ *and set* $k = 1$.

Step 3. *Compute the descent direction* w_n^k *of* J *at* u_n^k, $w_n^k = -\nabla J(u_n^k)$.

Step 4. *If* $\|w_n^k\| < \epsilon$, *then stop and output* u_n^k. *Otherwise, go to Step 5.*

Step 5. *Let* $v_n^k(s) = \frac{v_n^k + s w_n^k}{\|v_n^k + s w_n^k\|}$ *and solve for*

$$p(v_n^k(s)) = \sum_{i=1}^{n-1} t_i^k u_i + t_n^k v_n^k(s) = \arg \max_{t_i\in R, i=1,\dots,n} J(\sum_{i=1}^{n-1} t_i u_i + t_n v_n^k(s)).$$

with initial guess $(t_1^k, t_2^k, ..., t_n^k)$. *Set*

$$s_n^k = \max\{s | \lambda \geq s\|w_n^k\| \geq 0, J(p(v_n^k(s))) - J(p(v_n^k)) \leq -\frac{1}{2} s t_n^k \|w_n^k\|^2\}.$$

Let $v_n^{k+1} = v_n^k(s_n^k) = \frac{v_n^k + s_n^k w_n^k}{\|v_n^k + s_n^k w_n^k\|}$ *and* $u_n^{k+1} = p(v_n^{k+1}) = \sum_{i=1}^{n-1} t_i^{k+1} u_i + t_n^{k+1} v_n^{k+1}$. $k = k+1$ *and go to Step 3.* ▮

CHAPTER II

A LOCAL MIN-ORTHOGONAL METHOD

In this chapter, we establish a general framework for characterizing coexisting saddle points by modifying the original framework of a local minimax method developed in [38,39]. We first propose a new local L-$\perp$ selection based on which a solution manifold $\mathcal{M}$ can be defined. We then establish a local min-orthogonal characterization for coexisting saddle points and develop a local min-orthogonal method (LMOM) as well. Finally, a subsequence convergence of the algorithm is given.

A. Motivation

Consider cooperative elliptic systems of the form

$$\begin{cases} -\Delta u(x) = G_u(x, u(x), v(x)), & x \in \Omega, \\ -\Delta v(x) = G_v(x, u(x), v(x)), & x \in \Omega, \\ u(x) = v(x) = 0\ , & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$; $G : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 in the second and third variables, satisfying the following hypotheses

(A1) $|\nabla G(x,z)| \leq C(1 + |z|^{p-1})$, $\quad \forall z \in \mathbb{R}^2$, a.e. $x \in \Omega$, for some constants $C > 0$ and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$ or $2 < p < +\infty$ if $N = 1, 2$ (subcritical growth [61, 64]),

(A2) $G_u(x, 0, v(x)) \equiv G_v(x, u(x), 0) \equiv 0$ (homogeneity),

(A3) when $|(u,v)| \to 0$, $G(x,u,v) = \frac{\alpha}{2}u^2 + \frac{\beta}{2}v^2 + o(|(u,v)|^2)$ for some constants α, β s.t. $\max\{\alpha, \beta\} < \sigma_1$, where (G_u, G_v) is the gradient of G in the variables (u, v), σ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Condition (A1) is imposed in order to apply the continuous embedding $W^{1,q}(\Omega) \subset L^{q^*}(\Omega)$, where $q^* = qN/(N-q) > q$ ($q^* = \infty$, if $q \leq N$). Here, $q = 2$.

Condition (A1) implies that weak solutions of (2.1) are precisely critical points of the C^1-functional $J: \ H_0^1(\Omega) \times H_0^1(\Omega) \ \to \mathbb{R}$ given by

$$J(u, v) = \frac{1}{2} \int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2)\, dx - \int_\Omega G(x, u(x), v(x))\, dx. \tag{2.2}$$

It can be easily verified that $(0, 0)$ is a local minimum of such J under (A2) and (A3). The existence of multiple nontrivial solutions to system (2.1) has been established for several subclasses [6,46,64,65,68]. Among those nontrivial solutions, solutions without a zero component are of particular interest in many applications [27,34,52] since they represent the coexisting states which usually give rise to rather complex scenarios. Thus, we are only interested in finding such coexisting states (solutions) to (2.1) and shall view all other solutions as trivial. In addition, hypothesis (A2) implies that if one component (i.e., u or v) is equal to zero, then system (2.1) reduces to a semilinear elliptic equation w.r.t. the other component, for which many results (see, e.g., [48,57,61]) on the existence of multiple or infinitely many solutions can be applied.

Let us now recall the framework [38,39] of LMM in search for saddle points. For a given Hilbert space H, one first sets or selects a subspace $L \subset H$, also called a support, which is spanned by all trivial and known solutions at lower critical levels and from which an algorithm search needs to keep away; then introduces a composite functional $J(p(u))$ such that $u \in L^\perp$, $p(u) \in H \setminus L$ and $J'(p(u)) \perp \text{span}\{L, u\}$; and finally seeks a point $u^* \in L^\perp$ such that $J'(p(u^*)) \perp L^\perp$, which implies that $p(u^*) \notin L$ is a critical point of J. For example, if $p(u)$ is a local maximum of J on $\text{span}\{L, u\}$, then $J'(p(u)) \perp \text{span}\{L, u\}$; on the other hand, if a point $u^* \in L^\perp$ is a local minimizer of $J(p(u))$ on $L^\perp$, then $J'(p(u^*))\perp L^\perp$. In this case, one can immediately obtain $J'(p(u^*)) = 0$ since $J'(p(u^*))\perp(L^\perp \oplus L) \equiv H$.

When it comes to solving cooperative systems (2.1) for coexisting solutions, there may exist many and even infinitely many trivial (i.e., non-coexisting) solutions that need to be excluded. Under the framework of LMM, the support L may need to include so many trivial solutions at various critical levels. This can cause serious problems in numerical implementation. In this chapter, we shall fix this problem by introducing a new L-orthogonal selection and then establishing a new local characterization for coexisting saddle points.

B. Local Characterization on Coexisting Solutions

For $i = 1, 2$, let H_i be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, L_i be a closed subspace of H_i and $H_i = L_i \oplus L_i^\perp$ be its orthogonal decomposition. Denote $H = H_1 \times H_2$, $L = L_1 \times L_2$. Then, $L^\perp = L_1^\perp \times L_2^\perp$ and $H = L \oplus L^\perp$. Occasionally, L is called a "support" and L_1, L_2 are called "sub-supports". Denote by $S_B = \{u \in B : \|u\| = 1\}$ the unit sphere of any closed nonzero subspace B of H_i $(i = 1, 2)$ or H and let $[L_i, v] = \{tv + w | w \in L_i, t \in \mathbb{R}\}$, $\forall v \in S_{L_i^\perp}, i = 1, 2$. Assume $J \in C^1(H, \mathbb{R})$ and denote its gradient by $J' \equiv (\partial J_1, \partial J_2)$.

Definition II.1 ([16]) *A set-valued mapping P: $S_{L^\perp} \to 2^H$ is called an L-$\perp$ mapping of J if*

$$P(w) = \Big\{u \in [L_1, w_1] \times [L_2, w_2] : \partial J_i(u) \perp [L_i, w_i], i = 1, 2\Big\}, \quad \forall w = (w_1, w_2) \in S_{L^\perp}.$$

A single-valued mapping $p : S_{L^\perp} \to H$ is an L-$\perp$ selection of J if $p(w) \in P(w)$, $\forall w \in S_{L^\perp}$. For each $w \in S_{L^\perp}$, if p is locally defined around w, then p is called a local L-$\perp$ selection of J w.r.t. L at w. In particular, if $p(w)$ is a local maximum point of

J in $[L_1, w_1] \times [L_2, w_2]$, then p is called a local peak selection of J w.r.t. L at w.

Remark II.1 (a) Definition II.1 is stronger than the original one proposed in [66] (see also Definition IV.1) since

$$\partial J_1 \perp [L_1, w_1],\ \partial J_2 \perp [L_2, w_2] \Rightarrow J' = (\partial J_1, \partial J_2) \perp [L, w],\ \forall w = (w_1, w_2).$$

This new definition, however, not only enables us to identify and capture the coexisting solutions, but also allows us to deal with other types of elliptic systems such as noncooperative elliptic systems, refer to those problems studied in Chapter V.B.

(b) In view of systems (2.1), Definition II.1 holds some advantages for finding the coexisting solutions. Consider the case in which $\dim(L)$ is zero, for example. Assume $w = (w_1, w_2) \in S_{L^\perp}$ is our initial search direction such that $w_1 \neq 0, w_2 \neq 0$. Then, by Definition II.1, to find a peak selection p of J at w, we perform a 2-dimension search, picking up a local maximum of J in the 2-dimension subspace $\text{span}\{w_1\} \times \text{span}\{w_2\}$. If the gradient of J at that local maximum does not vanish, we then update our search direction with a point close to w via certain stepsize rule and perform a 2-dimension search again. We continue this process until a solution is found. It turns out that the repeated 2-dimension search in computing an L-$\perp$ selection will help us to stay away from those non-coexisting solutions eventually. Meanwhile, under the original definition, a peak selection p of J at w is just a local maximum of J along the direction w, which apparently is an outcome of a 1-dimension search. This 1-dimension search can assist us to keep away from the origin $(0, 0)$ but not from those non-coexisting solutions. Therefore, this new definition is more helpful to us in search for coexisting solutions.

(c) One can easily extend Definition II.1 to an n-product Hilbert space so that an n-component system can be solved, see Definition IV.2 and Section V.A.2.

It is easy to see the orthogonality in Definition II.1 is preserved under the limiting process in $S_{L^\perp}$. This leads to the following lemma.

Lemma II.1 *The graph $G = \{(u,w) : w \in S_{L^\perp}, u \in P(w) \neq \emptyset\}$ is closed.*

Proof. Let $(u^{(n)}, w^{(n)}) \in G$ and $(u^{(n)}, w^{(n)}) \to (u^{(0)}, w^{(0)})$. Since $w^{(n)} \to w^{(0)}$ and $S_{L^\perp}$ is closed, we have $w^{(0)} \in S_{L^\perp}$. If denoting $w^{(n)} = (w_1^{(n)}, w_2^{(n)})$, by definition,

$$u^{(n)} \in [L_1, w_1^{(n)}] \times [L_2, w_2^{(n)}] \quad \text{and} \quad \partial J_1(u^{(n)}) \perp [L_1, w_1^{(n)}],\ \partial J_2(u^{(n)}) \perp [L_2, w_2^{(n)}],$$

where $J'(u^{(n)}) = \big(\partial J_1(u^{(n)}), \partial J_2(u^{(n)})\big)$ and $u^{(n)} = (t_1^{(n)} w_1^{(n)}, t_2^{(n)} w_2^{(n)}) + (w_{L_1}^{(n)}, w_{L_2}^{(n)})$ for some $(t_1^{(n)}, t_2^{(n)}) \in \mathbb{R}^2$ and some $(w_{L_1}^{(n)}, w_{L_2}^{(n)}) \in L$. Denote $u^{(0)} = (u_{L_1}^\perp, u_{L_2}^\perp) + (u_{L_1}^{(0)}, u_{L_2}^{(0)})$, where $(u_{L_1}^\perp, u_{L_2}^\perp) \in L^\perp$, $(u_{L_1}^{(0)}, u_{L_2}^{(0)}) \in L$. Since $w^{(n)} \to w^{(0)}$ and

$$\|u^{(n)} - u^{(0)}\|^2 = \|t_1^{(n)} w_1^{(n)} - u_{L_1}^\perp\|^2 + \|t_2^{(n)} w_2^{(n)} - u_{L_2}^\perp\|^2 + \|(w_{L_1}^{(n)}, w_{L_2}^{(n)}) - (u_{L_1}^{(0)}, u_{L_2}^{(0)})\|^2 \to 0,$$

we obtain $(t_1^{(n)} w_1^{(n)}, t_2^{(n)} w_2^{(n)}) \to (u_{L_1}^\perp, u_{L_2}^\perp) = (t_1^{(0)} w_1^{(0)}, t_2^{(0)} w_2^{(0)})$ for some $(t_1^{(0)}, t_2^{(0)})$. Thus, $u^{(0)} = (t_1^{(0)} w_1^{(0)}, t_2^{(0)} w_2^{(0)}) + (u_{L_1}^{(0)}, u_{L_2}^{(0)}) \in [L_1, w_1^{(0)}] \times [L_2, w_2^{(0)}]$ and $\partial J_1(u^{(0)}) \perp [L_1, w_1^{(0)}]$, $\partial J_2(u^{(0)}) \perp [L_2, w_2^{(0)}]$ since J is $\mathcal{C}^1$. Therefore $u^{(0)} \in P(w^{(0)})$, i.e., $(u^{(0)}, w^{(0)}) \in G$. ∎

With Definition II.1, we give a necessary and sufficient condition for characterizing a coexisting critical point of J as below.

Theorem II.1 *Let $w^* = (w_1^*, w_2^*) \in S_{L^\perp}$ and p be a local L-$\perp$ selection of J w.r.t. L at w^* and continuous at w^*. Assume that $p_1(w^*) \notin L_1$ and $p_2(w^*) \notin L_2$, where $p(w^*) = (p_1(w^*), p_2(w^*))$, then a necessary and sufficient condition that $u^* = p(w^*)$ is a coexisting critical point of J is that there exists a neighborhood $\mathcal{N}(w^*)$ of w^* s.t.*

$$\partial J_i(p(w^*)) \perp p_i(w) - p_i(w^*),\ \forall w \in \mathcal{N}(w^*) \cap S_{L^\perp}, \quad i = 1, 2, \tag{2.3}$$

where $J'(p(w^*)) = (\partial J_1(p(w^*)), \partial J_2(p(w^*)))$, $p(w) = (p_1(w), p_2(w))$.

Proof. Only need to prove the sufficiency. Since $\partial J_i(p(w^*)) \perp L_i$ $(i = 1,2)$, it suffices to show that $\partial J_i(p(w^*)) \perp L_i^{\perp}$ $(i = 1,2)$. Let $\mathcal{N}(w^*)$ be a neighborhood of w^* s.t. p is well-defined and (2.3) is satisfied. Denote $p_i(w^*) = t_i^* w_i^* + w_{L_i}^*$ for some scalar t_i^* and $w_{L_i}^* \in L_i$, $i = 1,2$. Then $p_1(w^*) \notin L_1$ and $p_2(w^*) \notin L_2$ imply $w_i^* \neq 0$ and $t_i^* \neq 0$, $i = 1,2$. By the continuity, $p_i(w) = t_i w_i + w_{L_i}$ with $t_i \neq 0$ and $w_{L_i} \in L_i$ $(i = 1,2)$ for each $w = (w_1, w_2) \in \mathcal{N}(w^*) \cap S_{L^{\perp}}$. For $i = 1,2$, since $\partial J_i(p(w^*)) \perp [L_i, w_i^*]$ and $p_i(w^*) \in [L_i, w_i^*]$, we have

$$\partial J_i(p(w^*)) \perp p_i(w) - p_i(w^*) \Leftrightarrow \ \partial J_i(p(w^*)) \perp p_i(w) \Leftrightarrow \partial J_i(p(w^*)) \perp w_i, \qquad (2.4)$$

for each $w = (w_1, w_2) \in \mathcal{N}(w^*) \cap S_{L^{\perp}}$. On the other hand, for each $w \in L^{\perp}$, when the scalar s is small, we have

$$\frac{w^* + sw}{\|w^* + sw\|} \in \mathcal{N}(w^*) \cap S_{L^{\perp}}.$$

This and (2.4) imply that $\partial J_i(p(w^*)) \perp w_i$ $(i = 1,2)$, $\forall w = (w_1, w_2) \in L^{\perp}$, or $\partial J_i(p(w^*)) \perp L_i^{\perp}$, $i = 1,2$. ∎

Lemma II.2 *Let v be any unit vector in a normed space $(X, \|\cdot\|)$, then*

$$\|v - \frac{v \pm w}{\|v \pm w\|}\| \leq \frac{2\|w\|}{\|v \pm w\|}, \forall w \in X \backslash \{v\}.$$

The following lemma is crucial in developing our local min-orthogonal method and will lead to a local characterization for coexisting critical points of dual functionals and a stepsize rule for our local min-orthogonal algorithm as well.

Lemma II.3 *Let $w = (w_1, w_2) \in S_{L^{\perp}}$ with $w_1 \neq 0$, $w_2 \neq 0$ and p be a continuous local L-$\perp$ selection of J w.r.t. L at w. Denote $p(w) = (p_1(w), p_2(w)) = (t_1 w_1, t_2 w_2) + w_L$*

for some scalars t_1, t_2 *and* $w_L \in L$. *If* $t_1 t_2 > 0$, *then either* $J'(p(w)) = (0,0)$ *or there exists* $s_0 > 0$ *s.t.*

$$\begin{aligned} J(p(w(s))) - J(p(w)) &< -\frac{\min\{|t_1|,|t_2|\}s\|J'(p(w))\|^2}{2\sqrt{1+s^2\|J'(p(w))\|^2}} \\ &\leq -\frac{1}{4}\min\{|t_1|,|t_2|\}\|J'(p(w))\|\cdot\|w(s)-w\| < 0, \quad (2.5) \end{aligned}$$

$\forall\, 0 < s < s_0$, *where* $w(s) = (w_1(s), w_2(s)) = \dfrac{w - sign(t_1)sJ'(p(w))}{\sqrt{1+s^2\|J'(p(w))\|^2}} \in S_{L^\perp}$ *and*

$$p(w(s)) = (p_1(w(s)), p_2(w(s))) = (t_1(s)w_1(s), t_2(s)w_2(s)) + w_L(s)$$

for some $w_L(s) \in L$.

Proof. For convenience, let $d = (d_1, d_2) = -J'(p(w))$. Since $H = (L_1 \times L_2) \oplus (L_1^\perp \times L_2^\perp)$ and $w(s) \to w$ as $s \to 0$, p is continuous at w implies that $\|p(w(s)) - p(w)\| \to 0$ and

$$t_1(s) \to t_1, \quad t_2(s) \to t_2, \quad \text{as } s \to 0. \tag{2.6}$$

With $t_1 t_2 > 0$ and $J \in C^1(H, \mathbb{R})$, we have

$$sign(t_1(s)) = sign(t_2(s)) = sign(t_1) = sign(t_2) \tag{2.7}$$

and

$$\begin{aligned} J(p(w(s))) - J(p(w)) &= \langle J'(p(w)), p(w(s)) - p(w)\rangle + o(\|p(w(s)) - p(w)\|) \quad (2.8) \\ = \langle -d_1, p_1(w(s)) - p_1(w)\rangle &+ \langle -d_2, p_2(w(s)) - p_2(w)\rangle + o(\|p(w(s)) - p(w)\|) \end{aligned}$$

when s is small. Next, we note that p is an L-$\perp$ selection, i.e., $d_1 \perp [L_1, w_1]$, $d_2 \perp [L_2, w_2]$. It then follows that $\langle d_1, p_1(w)\rangle = \langle d_2, p_2(w)\rangle = 0$. This together with (2.7) leads to

$$\langle J'(p(w)), p(w(s)) - p(w)\rangle = \sum_{i=1}^{2}\langle -d_i, p_i(w(s))\rangle$$

$$\begin{aligned}
&= -\sum_{i=1}^{2}\langle d_i, t_i(s)w_i(s)\rangle = -\sum_{i=1}^{2}\langle d_i, t_i(s)\frac{w_i + sign(t_i)sd_i}{\sqrt{1+s^2\|d\|^2}}\rangle \text{ (since } w_L(s)\in L)\\
&= -\sum_{i=1}^{2}\langle d_i, \frac{sign(t_i)t_i(s)sd_i}{\sqrt{1+s^2\|d\|^2}}\rangle = -\sum_{i=1}^{2}\frac{sign(t_i)t_i(s)s}{\sqrt{1+s^2\|d\|^2}}\|d_i\|^2 \text{ (since } d_i\perp[L_i, w_i])\\
&= -\sum_{i=1}^{2}\frac{sign(t_i(s))t_i(s)s}{\sqrt{1+s^2\|d\|^2}}\|d_i\|^2 = -\sum_{i=1}^{2}\frac{|t_i(s)|s\|d_i\|^2}{\sqrt{1+s^2\|d\|^2}} \qquad (2.9)\\
&\le -\frac{\min(|t_1(s)|, |t_2(s)|)s}{\sqrt{1+s^2\|d\|^2}}\|d\|^2,
\end{aligned}$$

when s is small. From (2.6), it follows that $|t_1(s)| \to |t_1| > 0$, $|t_2(s)| \to |t_2| > 0$ and $\min\{|t_1(s)|, |t_2(s)|\} \to \min\{|t_1|, |t_2|\} > 0$ as $s \to 0$. Hence, when s is small, $\min\{|t_1(s)|, |t_2(s)|\} > \frac{1}{2}\min\{|t_1|, |t_2|\} > 0$. From this and (2.9), there is $s_0 > 0$ s.t.

$$\langle J'(p(w)), p(w(s)) - p(w)\rangle \le -\frac{\min(|t_1(s)|, |t_2(s)|)s}{\sqrt{1+s^2\|d\|^2}}\|d\|^2 < -\frac{\min(|t_1|, |t_2|)s}{2\sqrt{1+s^2\|d\|^2}}\|d\|^2 \quad (2.10)$$

for $0 < s < s_0$. On the other hand, by Lemma II.2 and the orthogonality $d\perp w$,

$$\|w - w(s)\| = \|w - \frac{w + sign(t_1)sd}{\|w + sign(t_1)sd\|}\| \le \frac{2s\|d\|}{\|w + sign(t_1)sd\|} = \frac{2s\|d\|}{\sqrt{1+s^2\|d\|^2}}. \quad (2.11)$$

Combining (2.10) and (2.11), we obtain

$$\langle J'(p(w)), p(w(s)) - p(w)\rangle < -\frac{\min(|t_1|, |t_2|)s}{2\sqrt{1+s^2\|d\|^2}}\|d\|^2 \le -\frac{1}{4}\min\{|t_1|, |t_2|\}\|d\|\cdot\|w(s) - w\|,$$

which together with (2.8) yields (2.5). ∎

With Lemma II.3, we can now establish the following local min-orthogonal characterization for coexisting critical points.

Theorem II.2 *If $w = (w_1, w_2) \in S_{L^\perp}$ and $p(w) = (p_1(w), p_2(w))$ be a local L-$\perp$ selection of J at w s.t. (i) p is continuous at w, (ii) $p_1(w) \notin L_1$ and $p_2(w) \notin L_2$ and (iii) $w = \arg(loc)\min_{v\in S_{L^\perp}} J(p(v))$, then $p(w)$ is a coexisting critical point of J.*

Proof. Suppose $J'(p(w)) \neq 0$. By (ii), we have $p(w) = (p_1(w), p_2(w)) = (t_1 w_1, t_2 w_2) + w_L$ for some scalars $t_1 \neq 0$, $t_2 \neq 0$ and $w_L \in L$. There are two cases: either (a) $t_1 t_2 > 0$ or (b) $t_1 t_2 < 0$. Since Case (b) can be converted to Case (a) by setting $\tilde{w} = (w_1, -w_2)$, we only need to discuss Case (a). For Case (a), by Lemma II.3, there is $s_0 > 0$ s.t. for $0 < s < s_0$,

$$J(p(w(s))) < J(p(w)) - \frac{1}{4}\min\{|t_1|, |t_2|\}\|J'(p(w))\| \cdot \|w(s) - w\| < J(p(w)),$$

which contradicts (iii). ∎

Remark II.2 If introducing a solution manifold $\mathcal{M}$ by

$$\mathcal{M} = \Big\{p(w) : w \in S_{L^\perp}\Big\},$$

then Lemma II.1 shows that $\mathcal{M}$ is closed and Theorem II.2 states that a local minimizer of $J(\cdot)$ on $\mathcal{M}$ (or $J(p(\cdot))$ on $S_{L^\perp}$) yields a saddle point $p(w^*)$ of J, which can be numerically approximated by a minimization method, e.g., the steepest descent method. The subspace L here serves as a support in search for a local minimizer of $J(\cdot)$ outside L.

C. Local Min-Orthogonal Algorithm

In this section, we present the flow chart of a local min-orthogonal algorithm, based on the local min-orthogonal characterization for coexisting saddle points established in Theorem II.2. A subsequence convergence of the algorithm is given in the end of this section.

1. The Flow Chart

Assume $u_1, \ldots, u_m \in H_1$ are linearly independent, so are $v_1, ..., v_n \in H_2$. Let $L = L_1 \times L_2$, where $L_1 = span\{u_1, \ldots, u_m\}$, $L_2 = span\{v_1, \ldots, v_n\}$. Choose an error tolerance $\varepsilon > 0$ and a stepsize control parameter $\lambda \in (0,1)$.

Algorithm II.1 *Local Min-Orthogonal Algorithm (LMOA)*

Step 1: *Set $k = 1$. Choose a point $\theta^{(k)} = (\theta_1^{(k)}, \theta_2^{(k)}) \in S_{L^\perp}$ s.t. $\theta_1^{(k)} \neq 0, \theta_2^{(k)} \neq 0$ and an appropriate initial guess $(t_0^{(k)}, ..., t_m^{(k)}, r_0^{(k)}, ..., r_n^{(k)})$ usually with $t_0^{(k)} r_0^{(k)} \neq 0$. Use this initial guess to solve a system of $m+n+2$ nonlinear equations*

$$\frac{\partial J(p(\theta^{(k)}))}{\partial t_i^{(k)}} = 0, \quad \frac{\partial J(p(\theta^{(k)}))}{\partial r_j^{(k)}} = 0, \quad i = 0, 1, ..., m, j = 0, 1, ..., n, \tag{2.12}$$

for the $m+n+2$ unknowns $(t_0^{(k)}, ..., t_m^{(k)}, r_0^{(k)}, ..., r_n^{(k)})$; more precisely, find an L-$\perp$ selection $p(\theta^{(k)}) = (p_1(\theta^{(k)}), p_2(\theta^{(k)}))$ at $\theta^{(k)}$ s.t.

$$\begin{aligned} p_1(\theta^{(k)}) &= \textstyle\sum_{i=1}^m t_i^{(k)} u_i + t_0^{(k)} \theta_1^{(k)} \in [L_1, \theta_1^{(k)}] \backslash L_1, \\ p_2(\theta^{(k)}) &= \textstyle\sum_{i=1}^n r_i^{(k)} v_i + r_0^{(k)} \theta_2^{(k)} \in [L_2, \theta_2^{(k)}] \backslash L_2, \end{aligned} \qquad \text{and } \ t_0^{(k)} r_0^{(k)} > 0.$$

Step 2: *Set $w^{(k)} = p(\theta^{(k)})$ and compute the gradient $d^{(k)} = \left(d_1^{(k)}, d_2^{(k)}\right) = J'(w^{(k)})$.*

Step 3: *If $\|d^{(k)}\| < \varepsilon$, then OUTPUT $w^{(k)}$, STOP; Otherwise, GOTO Step 4.*

Step 4: *For each $s > 0$, let $\theta^{(k)}(s) = (\theta_1^{(k)}(s), \theta_2^{(k)}(s)) = \dfrac{\theta^{(k)} - sign(t_0^{(k)}) s d^{(k)}}{\|\theta^{(k)} - sign(t_0^{(k)}) s d^{(k)}\|}$. Determine the stepsize*

$$\begin{aligned} s^k = \max_{i \in \mathbb{N}} \Bigg\{ \frac{\lambda}{2^i} \Bigg| & 2^i > \|d^{(k)}\|, \\ & J(p(\theta^{(k)}(\frac{\lambda}{2^i}))) - J(w^{(k)}) < -\frac{1}{4} \min\{|t_0^{(k)}|, |r_0^{(k)}|\} \|d^{(k)}\| \cdot \|\theta^{(k)}(\frac{\lambda}{2^i}) - \theta^{(k)}\| \Bigg\}, \end{aligned}$$

where $(t_0^{(k)}, t_1^{(k)}, \ldots, t_m^{(k)}, r_0^{(k)}, r_1^{(k)}, \ldots, r_n^{(k)})$ is used as an initial guess to evaluate $p(\theta^{(k)}(\frac{\lambda}{2^i})) = (p_1(\theta^{(k)}(\frac{\lambda}{2^i})), p_2(\theta^{(k)}(\frac{\lambda}{2^i})))$ with $p_1(\theta^{(k)}(\frac{\lambda}{2^i})) \notin L_1$ and $p_2(\theta^{(k)}(\frac{\lambda}{2^i})) \notin$

L_2 in the same way as in Step 1.

Step 5: *Set $\theta^{(k+1)} = \theta^{(k)}(s^k), p(\theta^{(k+1)}) = p(\theta^{(k)}(s^k)), k \leftarrow k+1$. GOTO Step 2.* ∎

Remark II.3 **(a)** The algorithm generally starts from the case $L_1 = L_2 = \{0\}$. With different initial points in $S_{L^\perp}$, it can find one or more saddle points (solutions), say (u_1^1, v_1^1), $(u_1^2, v_1^2), \ldots, (u_1^{e_1}, v_1^{e_1})$. Gradually increasing the dimension of L_1 and/or L_2 (e.g., with the sub-support L_1 fixed in span$\{u_1^1, u_1^2, \ldots, u_1^{e_1}\}$, one can gradually increase the dimension of the other, i.e., L_2, by adding more and more linearly independent elements from span$\{v_1^1, v_1^2, \ldots, v_1^{e_1}\}$), it then can capture one or more saddle points $(u_2^1, v_2^1), (u_2^2, v_2^2), \ldots, (u_2^{e_2}, v_2^{e_2})$ at higher critical levels. By this way, multiple branches of solutions can be located. On the other hand, if a problem possesses certain symmetries, one may easily construct the sub-supports L_1 and L_2 by applying those symmetries, refer also to Section V.A.1.

(b) Equations (2.12) come exactly from the definition of an L-$\perp$ selection (i.e., $\partial J_1(p(\theta^{(k)})) \perp [L_1, \theta_1^{(k)}], \partial J_2(p(\theta^{(k)})) \perp [L_2, \theta_2^{(k)}]$) and are apparently satisfied for every critical point of J. To find a new critical point, one needs to choose an appropriate initial guess in Step 1 (e.g., except $t_0^{(1)}, r_0^{(1)}$, set all other entries $t_1^{(1)} = \ldots = t_m^{(1)} = r_1^{(1)} = \ldots = r_n^{(1)} = 0$), then use this initial guess to find a solution, still denoted by $(t_0^{(1)}, t_1^{(1)}, \ldots, t_m^{(1)}, r_0^{(1)}, r_1^{(1)}, \ldots, r_n^{(1)})$, of (2.12) s.t. $t_0^{(1)} \neq 0, r_0^{(1)} \neq 0$ (it hence implies $p_1(\theta^{(1)}) \notin L_1, p_2(\theta^{(1)}) \notin L_2$). The resulting solution will be saved in the computer memory and used as the initial guess in evaluating the next L-$\perp$ selection p. In Step 4, it is crucial to consistently follow the initial guess $(t_0^{(k)}, t_1^{(k)}, \ldots, t_m^{(k)}, r_0^{(k)}, r_1^{(k)}, \ldots, r_n^{(k)})$ when computing the L-$\perp$ selection p from equations (2.12). The purpose of such strategy is to prevent a possible oscillation of p between different solution branches in the solution manifold $\mathcal{M}$ and hence to keep such p "continuous". Note that equations (2.12) usually have multiple solutions when J possesses multiple critical points.

(c) The algorithm is stable in the sense that the energy functional J is strictly decreasing, i.e., $J(p(\theta^{(k+1)})) < J(p(\theta^{(k)}))$.

Finally, a subsequence convergence result similar to that of Theorems 3.1-3.2 in [39] reads as:

Theorem II.3 *Let L_i be a closed subspace of H_i, $i = 1, 2$, $L = L_1 \times L_2$. Assume that $J \in C^1(H_1 \times H_2, \mathbb{R})$ satisfies the (PS) condition and p is an L-$\perp$ selection of J w.r.t. L s.t. (i) p is continuous, (ii) dist$(p_i(\theta^{(k)}), L_i) > \alpha > 0$ $(i = 1, 2)$ for some $\alpha > 0$ and all $k = 1, 2, \ldots,$ and (iii) $\inf_{1 \leq k < \infty} J(p(\theta^{(k)})) > -\infty$, where $\{p(\theta^{(k)}) \equiv (p_1(\theta^{(k)}), p_2(\theta^{(k)}))\}$ is a sequence generated by Alg. II.1 (wherein the stopping condition $\|d^{(k)}\| < \varepsilon$ is replaced by $\|d^{(k)}\| = 0$). Then*

(a) $\{p(\theta^{(k)})\}$ possesses a subsequence converging to a coexisting critical point of J,

(b) any convergent subsequence of $\{p(\theta^{(k)})\}$ converges to a coexisting critical point of J.

Proof. Follows similar arguments in the proofs of Theorems 3.1-3.2 in [39] while applying condition (ii). ∎

Since Alg. II.1 is based on the steepest descent method, its rate of convergence is expected to be linear. To speed up the convergence, a Newton's method can be used after a number of iterations by Alg. II.1, refer also to [59].

2. Computation of the Gradient

In this section we instructively describe how to compute the gradient $\nabla J(w)$ and an L-$\perp$ selection $p(\theta)$ for the functional J in (2.2) and the associated system (2.1).

Let $\|\cdot\|$ be the norm of $H_1 = H_2 = H_0^1(\Omega)$ defined by the inner product $\langle u, v\rangle = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx,\ \forall u, v \in H_0^1(\Omega)$. Then for $H = H_1 \times H_2$, its inner product is $\langle w_1, w_2\rangle = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle$ and norm is $\|w_1\|^2 = \|u_1\|^2 + \|v_1\|^2$, where $w_1 = (u_1, v_1),\ w_2 = (u_2, v_2) \in H$. For J in (2.2), its Frechet derivative at $\bar{w} = (\bar{u}, \bar{v})$ is

$$J'(\bar{w}) = (-\Delta\bar{u} - G_u(x, \bar{u}, \bar{v}),\ -\Delta\bar{v} - G_v(x, \bar{u}, \bar{v})) \in W^{-1,2}(\Omega) \times W^{-1,2}(\Omega),$$

whose smoothness usually is very “poor”. Numerically, such $J'(\bar{w})$ may not be used as a search direction in H. However, we can use its canonical identification in H (also called the gradient of J at $\bar{w}$) defined by

$$\nabla J(\bar{w}) = \Delta^{-1}(-J'(\bar{w})) = \big(\bar{u} + \Delta^{-1}(G_u(x, \bar{u}, \bar{v})),\ \bar{v} + \Delta^{-1}(G_v(x, \bar{u}, \bar{v}))\big) \in H$$

as our search direction at $\bar{w}$. Obviously,

$$J'(\bar{w}) = (0, 0) \Leftrightarrow \nabla J(\bar{w}) = (0, 0) \Leftrightarrow \bar{w} \text{ is a critical point of } J.$$

If denoting $d = (d_1, d_2) = \nabla J(\bar{w})$, then one can solve it from the following linear elliptic system

$$\left\{\begin{array}{rcll} \Delta d_1(x) & = & \Delta\bar{u}(x) + G_u(x, \bar{u}(x), \bar{v}(x)), & x \in \Omega \\ \Delta d_2(x) & = & \Delta\bar{v}(x) + G_v(x, \bar{u}(x), \bar{v}(x)), & x \in \Omega \\ d_1(x) & = & d_2(x) = 0, & x \in \partial\Omega, \end{array}\right. \tag{2.13}$$

via any standard finite element solver, e.g., the MATLAB subroutine ASSEMPDE.

In Steps 1 and 4 of Alg. II.1, we need to compute $w = p(\theta)$, the value of a local L-$\perp$ selection p at a point $\theta = (\theta_1, \theta_2) \in S_{L_1^\perp \times L_2^\perp}$, where $L_1 = \text{span}\{u_1, ..., u_m\}$ and $L_2 = \text{span}\{v_1, ..., v_n\}$. From the definition of p, we may write

$$w = (w_1, w_2) = (t_0\theta_1 + \sum_{i=1}^{m} t_i u_i,\ r_0\theta_2 + \sum_{i=1}^{n} r_i v_i).$$

Here, the $m+n+2$ unknowns $t_0, t_1, ..., t_m, r_0, r_1, ..., r_n$ are solved from the orthogonal conditions

$$\partial J_1(w) \perp [L_1, \theta_1], \partial J_2(w) \perp [L_2, \theta_2]$$

which, through integration by parts, lead to a system of $m+n+2$ nonlinear algebraic equations

$$\left\{ \begin{array}{rcll} \displaystyle\int_\Omega \Big[\Delta w_1(x) + G_u(x, w_1(x), w_2(x))\Big] \theta_1 \, dx & = & 0, & \\ \displaystyle\int_\Omega \Big[\Delta w_1(x) + G_u(x, w_1(x), w_2(x))\Big] u_i \, dx & = & 0, & i = 1, \ldots, m, \\ \displaystyle\int_\Omega \Big[\Delta w_2(x) + G_v(x, w_1(x), w_2(x))\Big] \theta_2 \, dx & = & 0, & \\ \displaystyle\int_\Omega \Big[\Delta w_2(x) + G_v(x, w_1(x), w_2(x))\Big] v_j \, dx & = & 0, & j = 1, \ldots, n. \end{array} \right. \tag{2.14}$$

The system above then can be solved by a MATLAB subroutine FSOLVE or FMINUNC with an initial guess determined by the same strategy as described in Remark II.3(b).

CHAPTER III

A LOCAL MIN-MAX-ORTHOGONAL METHOD

This chapter is devoted to a local min-max-orthogonal method (LMMOM) for saddle points of strongly indefinite functionals whose Euler-Lagrange equations are elliptic systems of noncooperative type (1.9). A closer examination on the corresponding functionals J in (1.10) reveals that they are positive definite in u and negative definite in v, respectively. Using this crucial structure, we prove a local min-max-orthogonal characterization for saddle points of strongly indefinite functionals of form (1.10). We then develop a local min-max-orthogonal algorithm (LMMOA) by carefully devising a stepsize rule. Such stepsize rule plays an important role in establishing the subsequence convergence results for LMMOA.

A. Local Characterization on Saddle Points of Infinite MI

The lemma below gives some stronger results than that of Lemma II.2.

Lemma III.1 *For every unit vector w in a Hilbert space $(X, \|\cdot\|)$, there holds*

$$\frac{\|v\|}{\|w \pm v\|} \leq \left\| \frac{w \pm v}{\|w \pm v\|} - w \right\| \leq \frac{2\|v\|}{\|w \pm v\|}, \forall v \in X \ \text{ with } v \perp w.$$

For $i = 1, 2$, let H_i be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, L_i be a closed subspace of H_i and $H_i = L_i \oplus L_i^{\perp}$ be its orthogonal decomposition. Still, we denote $H = H_1 \times H_2$, $L = L_1 \times L_2$. Assume $J \in C^1(H, \mathbb{R})$ and denote its gradient by $\nabla J \equiv (\partial J_1, \partial J_2)$.

The following lemma is crucial in establishing a local characterization for saddle points of a class of strongly indefinite functionals J of form (1.10) and a stepsize rule for the LMMOM as well.

Lemma III.2 *Let* $J \in C^1(H, \mathbb{R})$, L *be a closed subspace of* H *and* $w = (w_1, w_2) \in S_{L^\perp}$ *with* $w_1 \neq 0$, $w_2 \neq 0$. *Assume that* p *is a continuous local* L-$\perp$ *selection of* J *w.r.t.* L *at* w. *Denote by* $d \equiv (d_1, d_2) = \nabla J(p(w))$ *the gradient of* J *at* $p(w)$ *and write* $p(w) \equiv (p_1(w), p_2(w)) = (t_1 w_1, t_2 w_2) + w_L \in H$, *where* $(t_1, t_2) \in \mathbb{R}^2$ *and* $w_L \in L$. *If* $t_1 t_2 \neq 0$ *and* $d \neq (0,0)$, *then* $\exists s_0 > 0$ *such that* $\forall 0 < s \leq s_0$ *there holds one of the followings*

(i) $J(p(w^{(1)}(s))) - J(p(w)) < -\frac{|t_1|}{4}||d_1|| \cdot ||w^{(1)}(s) - w|| \leq -\frac{|t_1|}{8}s||d_1||^2 < 0,$
if $d_2 = 0$*;*

(ii) $J(p(w)) - J(p(w^{(2)}(s))) < -\frac{|t_2|}{4}||d_2|| \cdot ||w^{(2)}(s) - w|| \leq -\frac{|t_2|}{8}s||d_2||^2 < 0,$
if $d_1 = 0$*;*

(iii) $$\begin{cases} J(p(w^{(1)}(s))) - J(p(w)) < -\frac{|t_1|}{4}||d_1|| \cdot ||w^{(1)}(s) - w|| \leq -\frac{|t_1|}{8}s||d_1||^2 < 0, \\ J(p(w)) - J(p(w^{(2)}(s))) < -\frac{|t_2|}{4}||d_2|| \cdot ||w^{(2)}(s) - w|| \leq -\frac{|t_2|}{8}s||d_2||^2 < 0, \end{cases}$$
if $d_i \neq 0\,(i = 1, 2)$*;*

where $w^{(1)}(s) = \dfrac{w - sign(t_1)s(d_1, 0)}{||w - sign(t_1)s(d_1, 0)||}$, $w^{(2)}(s) = \dfrac{w + sign(t_2)s(0, d_2)}{||w + sign(t_2)s(0, d_2)||} \in S_{L^\perp}$.

Proof. (i) Assume $d_1 \neq 0, d_2 = 0$. First, we note that p is an L-$\perp$ selection implies $d_1 \perp w_1$. Then we have

$$w^{(1)}(s) \equiv (w_1^{(1)}(s), w_2^{(1)}(s)) = (\frac{w_1 - sign(t_1)sd_1}{\sqrt{1 + s^2\|d_1\|^2}}, \frac{w_2}{\sqrt{1 + s^2\|d_1\|^2}}) \to w = (w_1, w_2)$$

as $s \to 0$. Since p is continuous at w, so $p(w^{(1)}(s)) \to p(w)$ when $s \to 0$. On the other hand, for each s near zero,

$$p(w^{(1)}(s)) \equiv (p_1(w^{(1)}(s)), p_2(w^{(1)}(s))) = (\tilde{t}_1(w^{(1)}(s))w_1^{(1)}(s), \tilde{t}_2(w^{(1)}(s))w_2^{(1)}(s)) + w_L(s)$$

for some scalars $\tilde{t}_1(w^{(1)}(s)), \tilde{t}_2(w^{(1)}(s))$ and some $w_L(s) \in L$. Thus $\tilde{t}_i(w^{(1)}(s)) \to t_i$ as $s \to 0, i = 1, 2$; in particular,

$$\tilde{t}_1(s) \to t_1 \quad \text{as } s \to 0 \tag{3.1}$$

if denoting $\tilde{t}_1(s) \equiv \tilde{t}_1(w^{(1)}(s))$.

With $t_1 t_2 \neq 0, d_2 = 0$ and $J \in C^1(H, \mathbb{R})$, we have

$$sign(\tilde{t}_1(s)) = sign(t_1), |\tilde{t}_1(s)| > |t_1|/2 \tag{3.2}$$

and

$$\begin{aligned} & J(p(w^{(1)}(s))) - J(p(w)) \\ = \ & \langle \nabla J(p(w)), p(w^{(1)}(s)) - p(w) \rangle + o(\|p(w^{(1)}(s)) - p(w)\|) \\ = \ & \textstyle\sum_{i=1}^2 \langle d_i, p_i(w^{(i)}(s)) - p_i(w) \rangle + o(\|p(w^{(1)}(s)) - p(w)\|) \\ = \ & \langle d_1, p_1(w^{(1)}(s)) - p_1(w) \rangle + o(\|p(w^{(1)}(s)) - p(w)\|) \ \ (\text{because } d_2 = 0) \end{aligned} \tag{3.3}$$

when s is small. Again, p is an L-$\perp$ selection implies $d_1 \perp [L_1, w_1]$, $d_2 \perp [L_2, w_2]$. It then follows that $\langle d_1, p_1(w) \rangle = \langle d_2, p_2(w) \rangle = 0$. Hence

$$\begin{aligned} & \langle \nabla J(p(w)), p(w^{(1)}(s)) - p(w) \rangle = \langle d_1, p_1(w^{(1)}(s)) \rangle \quad (\text{because } d_2 = 0) \\ = \ & \langle d_1, \tilde{t}_1(s) w_1^{(1)}(s) \rangle = \langle d_1, \tilde{t}_1(s) \frac{w_1 - sign(t_1) s d_1}{\sqrt{1 + s^2 \|d_1\|^2}} \rangle \quad (\text{because } w_L(s) \in L) \\ = \ & \langle d_1, \frac{-sign(t_1)\tilde{t}_1(s) s d_1}{\sqrt{1 + s^2 \|d_1\|^2}} \rangle = -\frac{sign(t_1)\tilde{t}_1(s) s}{\sqrt{1 + s^2 \|d_1\|^2}} \|d_1\|^2 \quad (\text{because } d_1 \perp [L_1, w_1]) \\ = \ & -\frac{sign(\tilde{t}_1(s))\tilde{t}_1(s) s}{\sqrt{1 + s^2 \|d_1\|^2}} \|d_1\|^2 = -\frac{|\tilde{t}_1(s)| s}{\sqrt{1 + s^2 \|d_1\|^2}} \|d_1\|^2 \\ < \ & -\frac{|t_1| s}{2\sqrt{1 + s^2 \|d_1\|^2}} \|d_1\|^2 \end{aligned} \tag{3.4}$$

for any $s > 0$.

By Lemma III.1, we have

$$\frac{1}{2}s\|d_1\| \leq \frac{s\|d_1\|}{\sqrt{1+s^2\|d_1\|^2}} \leq \left\|\frac{w - sign(t_1)s(d_1,0)}{\|w - sign(t_1)s(d_1,0)\|} - w\right\| \leq \frac{2s\|d_1\|}{\sqrt{1+s^2\|d_1\|^2}} \tag{3.5}$$

when $s > 0$ is sufficiently small. Combining (3.4) and (3.5) yields

$$\begin{aligned}\langle \nabla J(p(w)), p(w^{(1)}(s)) - p(w)\rangle &< -\tfrac{|t_1|}{4}\|d_1\|\tfrac{2s\|d_1\|}{\sqrt{1+s^2\|d_1\|^2}} \\ &\leq -\tfrac{|t_1|}{4}\|d_1\|\left\|\tfrac{w-sign(t_1)s(d_1,0)}{\|w-sign(t_1)s(d_1,0)\|} - w\right\| \\ &= -\tfrac{|t_1|}{4}\|d_1\| \cdot \|w^{(1)}(s) - w\| \leq -\tfrac{|t_1|}{8}s\|d_1\|^2\end{aligned}$$

when s is sufficiently small. Taking (3.3) into account, we conclude that $\exists s_0 > 0$ s.t. (i) holds $\forall 0 < s \leq s_0$. Finally, by similar arguments as above, one can prove (ii) and (iii). ∎

With the preceding lemma, we are ready to establish a local min-max-orthogonal characterization for saddle points of strongly indefinite functionals J of form (1.10).

Theorem III.1 *Let $J \in C^1(H,\mathbb{R})$ be a strongly indefinite functional of form (1.10), $\bar{w} = (\bar{w}_1, \bar{w}_2) \in S_{L^\perp}$. Assume that $p(\bar{w}) = (p_1(\bar{w}), p_2(\bar{w}))$ is a local L-$\perp$ selection of J w.r.t. L at $\bar{w}$ s.t. (i) p is continuous at $\bar{w}$, (ii) dist$(p_1(\bar{w}), L_1) > 0$ and dist$(p_2(\bar{w}), L_2) > 0$. If there exists an open neighborhood $U \times V \subset L^\perp$ of $(\bar{w}_1, \bar{w}_2)$ s.t.*

$$J(p(\tfrac{(\bar{w}_1,w_2)}{\|(\bar{w}_1,w_2)\|})) \leq J(p(\bar{w}_1,\bar{w}_2)) \leq J(p(\tfrac{(w_1,\bar{w}_2)}{\|(w_1,\bar{w}_2)\|})), \quad \forall (w_1,w_2) \in U \times V, \tag{3.6}$$

then $p(\bar{w})$ is a saddle point of J in H.

Proof. Only need to show that $p(\bar{w})$ is a critical point since every critical point of a strongly indefinite functional is a saddle point. Suppose, by contradiction, $\|\nabla J(p(\bar{w}))\| \equiv \|(d_1,d_2)\| \neq 0$. We have three cases: (a) $d_1 \neq 0, d_2 = 0$, (b) $d_1 = 0, d_2 \neq 0$, (c) $d_1 \neq 0, d_2 \neq 0$. By definition, we may write $p(\bar{w}) = (p_1(\bar{w}), p_2(\bar{w})) =$

$(t_1\bar{w}_1, t_2\bar{w}_2)+w_L$ for some scalars t_1, t_2 and some $w_L \in L$. Then condition (ii) implies that $\bar{w}_1 \neq 0, \bar{w}_2 \neq 0$ and $t_1t_2 \neq 0$. For case (a), by Lemma III.2, there is $s_0 > 0$ s.t.

$$J(p(w^{(1)}(s))) < J(p(\bar{w})) - \frac{1}{4}|t_1|\|\nabla J(p(\bar{w}))\| \cdot \|w^{(1)}(s) - \bar{w}\| < J(p(\bar{w}))$$

$\forall 0 < s \leq s_0$, where

$$w^{(1)}(s) \equiv (w_1^{(1)}(s), w_2^{(1)}(s)) = \frac{\bar{w} - sign(t_1)s(d_1, 0)}{||\bar{w} - sign(t_1)s(d_1, 0)||} = \frac{(\bar{w}_1 - sign(t_1)sd_1, \bar{w}_2)}{\sqrt{1 + s^2\|d_1\|^2}}.$$

This violates (3.6) when s is small. Finally, cases (b) and (c) can follow similar lines as does case (a). ∎

Remark III.1 As before, a solution manifold $\mathcal{M} \subset H$ can be defined by

$$\mathcal{M} = \Big\{p(w) : w \in S_{L^\perp}\Big\}.$$

Note that a point $\bar{w} \in S_{L^\perp}$ characterized by (3.6) is similar to a saddle point defined in game theory. Hence, we may call such $\bar{w}$ a game-type saddle point (or just saddle point if there is no confusion) of $J(p(\cdot))$ on $S_{L^\perp}$. Further, we say that $p(\bar{w})$ is a saddle point (actually a game-type saddle point) of J on $\mathcal{M}$. With this in mind, instead of finding saddle points of J in H, we actually look for saddle points of J on $\mathcal{M}$.

There are some variations of Lemma III.2 based on which slightly different step-size rules can be obtained. The following is one of such variations.

Lemma III.3 *Under the assumptions in Lemma III.2, there holds one of the followings*

(i) $J(p(w^{(1)}(s_1))) - J(p(w)) < -\frac{|t_1|}{4}||d_1|| \cdot ||w^{(1)}(s_1) - w|| \leq -\frac{|t_1|}{8}s_1||d_1||^2 < 0$, *if* $d_2 = 0$*;*

(ii) $J(p(w)) - J(p(w^{(2)}(s_2))) < -\frac{|t_2|}{4}||d_2|| \cdot ||w^{(2)}(s_2) - w|| \leq -\frac{|t_2|}{8}s_2||d_2||^2 < 0$, *if* $d_1 = 0$*;*

(iii) $$(-1)^i\Big(J(p(w))-J(p(w^{(i)}(s_i)))\Big) < -\tfrac{|t_i|}{4}||d_i||\cdot||w^{(i)}(s_i)-w|| \le -\tfrac{|t_i|}{8}s_i||d_i||^2<0, \qquad i=1,2,$$

if $d_1\neq 0, d_2\neq 0$;

$\forall 0<s_1\le\bar{s}_1, 0<s_2\le\bar{s}_2$, *for some* $\bar{s}_1,\bar{s}_2>0$, *where*

$$w^{(1)}(s_1)=\frac{w-sign(t_1)s_1(d_1,0)}{||w-sign(t_1)s_1(d_1,0)||}\in S_{L^\perp}, w^{(2)}(s_2)=\frac{w+sign(t_2)s_2(0,d_2)}{||w+sign(t_2)s_2(0,d_2)||}\in S_{L^\perp}.$$

Corollary III.1 *With the notations and assumptions in Lemma III.2, if letting* $j=\arg\max_{k\in\{1,2\}}\|d_k\|^2$, *then there exists* $\bar{s}>0$ *such that*

$$J(p(w^{(2)}(s)))-J(p(w^{(1)}(s)))>\frac{|t_j|}{8}s||d_j||^2\ge\frac{|t_j|}{16}s||d||^2$$

$\forall 0<s\le\bar{s}$.

Proof. Follows from Lemma III.2 and the fact $2||d_j||^2\ge\|d\|^2=\|d_1\|^2+\|d_2\|^2$. ∎

B. Local Min-Max-Orthogonal Algorithm

Algorithm III.1 *Local Min-Max-Orthogonal Algorithm (LMMOA)*

Step 0: *Set a support* $L=L_1\times L_2=span\{u_1,...,u_m\}\times span\{v_1,...,v_n\}$ *and a tolerance* $\epsilon>0$ *and choose control parameters* λ, T *s.t.* $0<\lambda<1, T\ge 1$.

Step 1: *Choose an initial direction* $w^1=(w_1^1,w_2^1)\in S_{L_1^\perp\times L_2^\perp}$ *with* $w_1^1\neq 0, w_2^1\neq 0$ *s.t.*

$$p_1(w^1)=\textstyle\sum_{i=1}^m t_i^1u_i+t_0^1w_1^1\in[L_1,w_1^1]\backslash L_1 \quad (i.e.,\ t_0^1\neq 0),$$
$$p_2(w^1)=\textstyle\sum_{i=1}^n r_i^1v_i+r_0^1w_2^1\in[L_2,w_2^1]\backslash L_2 \quad (i.e.,\ r_0^1\neq 0),$$

where $p(w^1)=(p_1(w^1),p_2(w^1))$ *is an* L-$\perp$ *selection of* J *w.r.t.* L *at* w^1 *and* t_i^1, r_j^1 $(i=0,\dots,m, j=0,\dots,n)$ *are solved from the following* $(m+n+2)$

equations

$$\partial J_1(p(w^1)) \perp w_1^1, \quad \partial J_1(p(w^1)) \perp u_i, \quad i = 1, \ldots, m,$$

$$\partial J_2(p(w^1)) \perp w_2^1, \quad \partial J_2(p(w^1)) \perp v_j, \quad j = 1, \ldots, n.$$

Set k=1.

Step 2: *Set* $\theta^k = p(w^k)$ *and compute the gradient* $d^k = (d_1^k, d_2^k) = \nabla J(\theta^k)$.

Step 3: *If* $\max\{||d_1^k||, ||d_2^k||\} \le \epsilon$, *OUTPUT* θ^k, *STOP; otherwise, GOTO Step 4.*

Step 4 (Find next point by the stepsize rule) :

Find $w^{k+1} \equiv (w_1^{k+1}, w_2^{k+1}) = \phi(\bar{s}_1, \bar{s}_2)$ *where*

$$\phi(s_1, s_2) \equiv \frac{(w_1^k(s_1), w_2^k(s_2))}{\|(w_1^k(s_1), w_2^k(s_2))\|} \tag{3.7}$$

with $w_1^k(s_1) = w_1^k - sign(t_0^k) s_1 d_1^k$, $w_2^k(s_2) = w_2^k + sign(r_0^k) s_2 d_2^k$, *and* $\bar{s}_1, \bar{s}_2$ *are determined by the following* **stepsize rule**.

(i) *First, initialize the stepsizes* $\bar{s}_1 = \bar{s}_2 = 0$.

(ii) *If* $||d_1^k|| > \epsilon$, *then*

$$\bar{s}_1 = \max_{i \in N} \left\{ \frac{\lambda}{2^i} \middle| 2^i > ||d_1^k||, \right.$$
$$\left. J(p(\phi(\tfrac{\lambda}{2^i}, 0))) - J(p(w^k)) < -\tfrac{|t_0^k|}{4} ||d_1^k|| \cdot ||\phi(\tfrac{\lambda}{2^i}, 0) - w^k|| \right\};$$

If $||d_2^k|| > \epsilon$, *then*

$$\bar{s}_2 = \max_{i \in N} \left\{ \frac{\lambda}{2^i} \middle| 2^i > ||d_2^k||, \right.$$
$$\left. J(p(w^k)) - J(p(\phi(0, \tfrac{\lambda}{2^i}))) < -\tfrac{|r_0^k|}{4} ||d_2^k|| \cdot ||\phi(0, \tfrac{\lambda}{2^i}) - w^k|| \right\}.$$

Here $(t_0^k, t_1^k, \ldots, t_m^k, r_0^k, r_1^k, \ldots, r_n^k)$ *is used as an initial guess to evaluate* $p(\phi(\frac{\lambda}{2^i}, 0))$ *and/or* $p(\phi(0, \frac{\lambda}{2^i}))$ *through the same way as does Step 1.*

(iii) (Adjust the stepsizes)

If $||d_1^k|| \le ||d_2^k||$, *set* $\bar{s}_1 = \min\{\bar{s}_1, T\bar{s}_2\}$; *otherwise, set* $\bar{s}_2 = \min\{\bar{s}_2, T\bar{s}_1\}$.

Step 5: *Evaluate* $p(w^{k+1})$ *and set* $k \leftarrow k+1$, *GOTO Step 2.* ∎

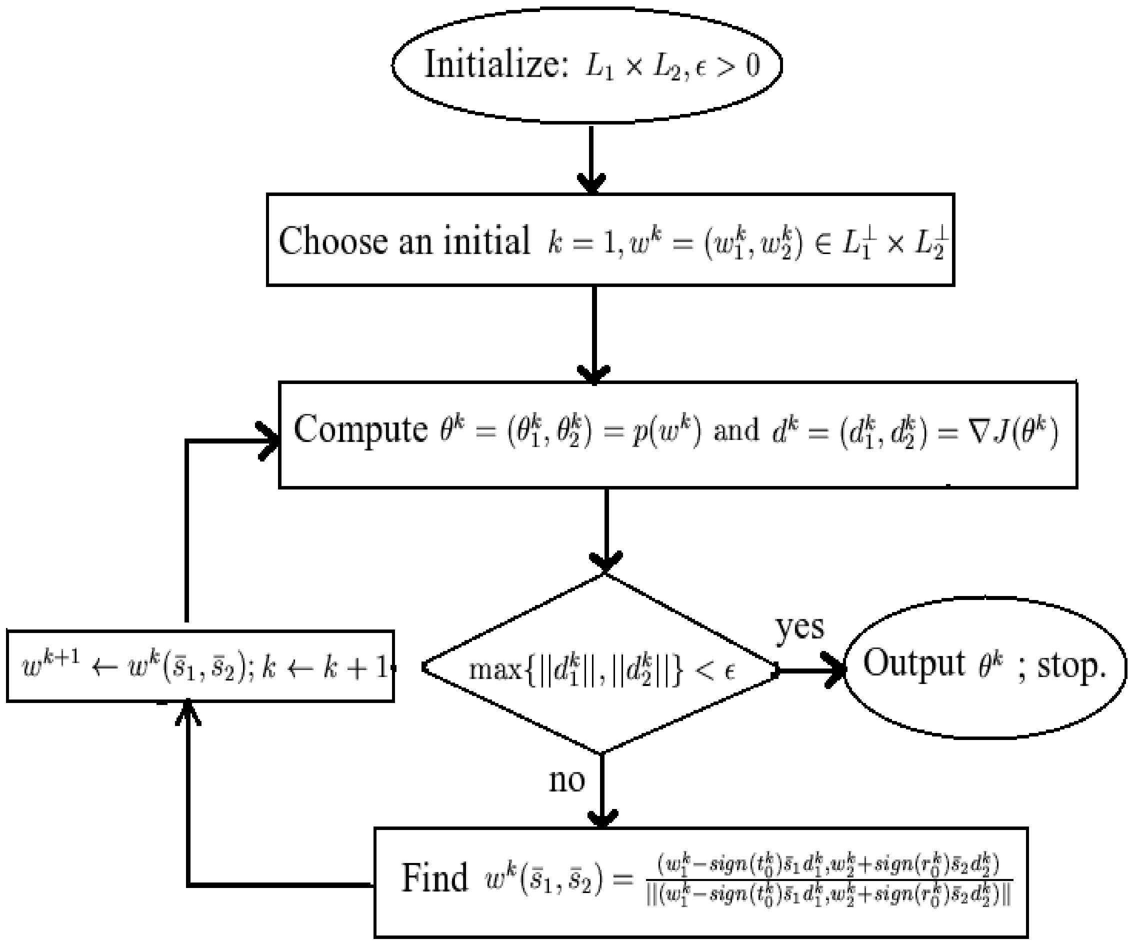

Fig. 1. Flow chart of the local min-max-orthogonal algorithm.

Remark III.2 A flow chart of Alg. III.1 is shown in Fig. 1 wherein the stepsizes $\bar{s}_1, \bar{s}_2$ are determined by Step 4 of Alg. III.1 and satisfy the following

$$\bar{s}_1 \begin{cases} > 0, & \text{if } \|d_1^k\| > \epsilon \\ = 0, & \text{if } \|d_1^k\| \leq \epsilon \end{cases}, \qquad \bar{s}_2 \begin{cases} > 0, & \text{if } \|d_2^k\| > \epsilon \\ = 0, & \text{if } \|d_2^k\| \leq \epsilon \end{cases}. \tag{3.8}$$

Due to (3.8), Alg. III.1 produces two byproducts, denoted by $\{w^{k,1}\}$ and $\{w^{k,2}\}$ (see also Fig. 2), such that

$$w^{k,1} = \begin{cases} \phi(\bar{s}_1, 0), & \text{if } \|d_1^k\| > \epsilon \\ w^k, & \text{if } \|d_1^k\| \leq \epsilon \end{cases}, \qquad w^{k,2} = \begin{cases} \phi(0, \bar{s}_2), & \text{if } \|d_2^k\| > \epsilon \\ w^k, & \text{if } \|d_2^k\| \leq \epsilon \end{cases}. \tag{3.9}$$

These two byproducts are actually very important for us to establish the convergence of the algorithm.

From the stepsize rule in Step 4 of Alg. III.1, one can easily see the following proposition.

Proposition III.1 $\quad J(p(w^{k,2})) \geq J(p(w^k)) \geq J(p(w^{k,1})), \ \forall k.$

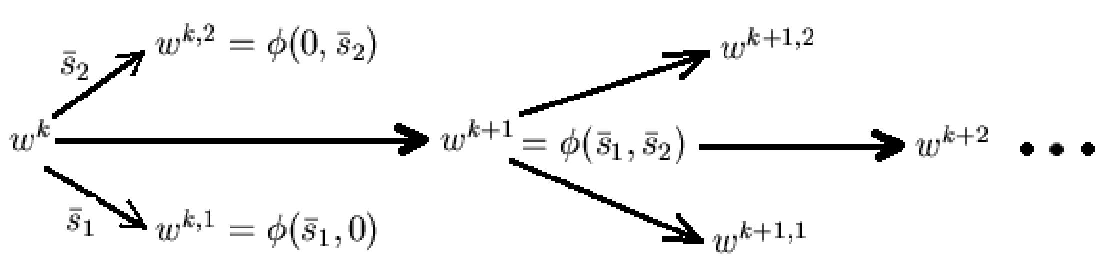

Fig. 2. Sequences $\{w^k\}, \{w^{k,1}\}, \{w^{k,2}\}$ generated by LMMOA.

To obtain the convergence of Alg. III.1, we need to slightly modify its stopping condition in Step 3; in other words, we assume $\{w^k \equiv (w_1^k, w_2^k)\}, \{w^{k,1}\}, \{w^{k,2}\}$ are infinite sequences generated by Alg. III.1 in the sense that the tolerance $\epsilon \to 0$. We continue to use the same notations as in Alg. III.1, i.e., $\theta^k = p(w^k)$, $d^k \equiv (d_1^k, d_2^k) = \nabla J(\theta^k)$.

Theorem III.2 (Uniform stepsize rule) *Suppose that $J \in C^1(H, \mathbb{R})$ and $L = L_1 \times L_2$ is a closed subspace of H. Let $\bar{w} \equiv (\bar{w}_1, \bar{w}_2) \in S_{L^\perp}$, $p(\bar{w}) \equiv (p_1(\bar{w}), p_2(\bar{w})) = (t_1\bar{w}_1, t_2\bar{w}_2) + w_L$ with $w_L \in L$ be a local L-$\perp$ selection of J w.r.t. L at $\bar{w}$ s.t. (i) p is continuous at $\bar{w}$, (ii) dist$(p_i(\bar{w}), L_i) > 0$ $(i = 1, 2)$. Denote by $\bar{d} \equiv (\bar{d}_1, \bar{d}_2) = \nabla J(p(\bar{w}))$ the gradient of J at $p(\bar{w})$. If $\|\bar{d}\| \neq 0$, then there holds one of the followings:*

(i) if $\bar{d}_2 = 0$, then there exists an open neighborhood U of $\bar{w}_1$ and a number $\bar{s}_1 > 0$ such that the stepsize rule in Lemma III.3(i) holds true for any $w_1 \in U \cap L_1^\perp$, wherein w is replaced by $\widetilde{w} \equiv \frac{(w_1, \bar{w}_2)}{\|(w_1, \bar{w}_2)\|} \in S_{L^\perp}$ and $d = (d_1, d_2)$ is replaced by $\nabla J(p(\widetilde{w}))$;

(ii) if $\bar{d}_1 = 0$, then there exists an open neighborhood V of $\bar{w}_2$ and a number $\bar{s}_2 > 0$ such that the stepsize rule in Lemma III.3(ii) holds true for any $w_2 \in V \cap L_2^\perp$, wherein w is replaced by $\widetilde{w} \equiv \frac{(\bar{w}_1, w_2)}{\|(\bar{w}_1, w_2)\|} \in S_{L^\perp}$ and $d = (d_1, d_2)$ is replaced by $\nabla J(p(\widetilde{w}))$;

(iii) if $\bar{d}_1 \neq 0, \bar{d}_2 \neq 0$, then there exists an open neighborhood $U \times V$ of $(\bar{w}_1, \bar{w}_2)$ and numbers $\bar{s}_1, \bar{s}_2 > 0$ such that the stepsize rule in Lemma III.3(iii) holds true for any $w = (w_1, w_2) \in (U \times V) \cap S_{L^\perp}$.

Proof. The proof is similar to that of Lemma 2.2 in [39]. ∎

Remark III.3 The preceding theorem states that for a given point $\bar{w} \equiv (\bar{w}_1, \bar{w}_2) \in S_{L^\perp}$ s.t. $\bar{w}_1 \neq 0, \bar{w}_2 \neq 0$, there exists a uniform stepsize $\bar{s}_1$ (resp. $\bar{s}_2$) locally independent of $\bar{w}_1$ (resp. $\bar{w}_2$) if the corresponding partial gradient d_1 (resp. d_2) is not zero. More precisely, if $d_1 \neq 0, d_2 = 0$, for example, then there is a uniform stepsize

$\bar{s}_1$ for any $w_1 \in U \cap L_1^\perp$, where U is some neighborhood of $\bar{w}_1$ (which hence implies that $\widetilde{w} \equiv \frac{(w_1,\bar{w}_2)}{\|(w_1,\bar{w}_2)\|} \in S_{L^\perp}$ is in some neighborhood of $\bar{w}$). In this case, however, the stepsize $\bar{s}_2$ may or may not be bounded below away from 0, i.e., $\bar{s}_2$ may decrease to 0 as $w_1 \to \bar{w}_1$ or equivalently $\widetilde{w} \to \bar{w} = (\bar{w}_1, \bar{w}_2)$. Finally, one sees that the adjustment of the stepsizes in Step 4(iii) of Alg. III.1 will not destroy such uniformity.

C. Subsequence Convergence

In this section we prove some convergence results for LMMOA. First, we show that under some appropriate assumptions any limit point of the sequence $\{w^k\}$ generated by Alg. III.1 will yield a critical point of the functional J in H. Next, we establish the existence of a subsequence of $\{w^k\}$ which leads to a critical point of J in H. The proof is based on an auxiliary lemma on estimating the distance of two consecutive iteration points, i.e., $\|w^{k+1} - w^k\|$.

Theorem III.3 *Suppose $J \in C^1(H_1 \times H_2, \mathbb{R})$, L_i is a closed subspace of H_i, $i = 1, 2$. Let $L = L_1 \times L_2$, $\{w^k\}, \{w^{k,1}\}, \{w^{k,2}\}$ be the sequences produced by Alg. III.1 in the sense the tolerance $\epsilon \to 0$. Assume further that the L-$\perp$ selection p therein satisfies*

(i) p is continuous,

(ii) dist$(p_i(w^k), L_i) \geq \alpha > 0$ $(i = 1, 2)$ for some $\alpha > 0$, $\forall k$,

(iii) $J(p(w^{k,2})) - J(p(w^{k,1})) \to 0$ when $k \to \infty$.

Then for any convergent subsequence $w^{k_i} = (w_1^{k_i}, w_2^{k_i})$ of w^k with $w^{k_i} \to \bar{w}$ for some $\bar{w} = (\bar{w}_1, \bar{w}_2) \in S_{L^\perp}$, $p(\bar{w})$ is a coexisting critical point of J.

Proof. We still borrow the notations from Alg. III.1. Let w^{k_i} be any convergent subsequence of w^k. By definition, $p(\bar{w})$ and $p(w^{k_i})$ can be expressed as

$$\begin{aligned} p(\bar{w}) &= (\bar{t}_1\bar{w}_1, \bar{t}_2\bar{w}_2) + \bar{w}_L, && \text{for some } \bar{w}_L \in L, \\ p(w^{k_i}) &= (t_1^{k_i}w_1^{k_i}, t_2^{k_i}w_2^{k_i}) + w_L^{k_i}, && \text{for some } w_L^{k_i} \in L, \forall k_i. \end{aligned}$$

It follows from conditions (i) and (ii) that $t_l^{k_i} \to \bar{t}_l, w_l^{k_i} \to \bar{w}_l, l = 1, 2$. Clearly, $\bar{t}_l \neq 0, \bar{w}_l \neq 0$ $(l = 1, 2)$ because $\text{dist}(p_l(w^{k_i}), L_l) = \|t_l^{k_i}w_l^{k_i}\| \geq \alpha > 0$ and

$$\text{dist}(p_l(w^{k_i}), L_l) \to \text{dist}(p_l(\bar{w}), L_l) = \|\bar{t}_l\bar{w}_l\| \geq \alpha > 0,\ l = 1, 2.$$

Next, denote by $(d_1, d_2) \equiv \nabla J(p(\bar{w}))$ and by $d^{k_i} \equiv (d_1^{k_i}, d_2^{k_i}) = \nabla J(p(w^{k_i}))$ the gradient of J at $p(\bar{w})$ and $p(w^{k_i})$, respectively. Suppose by contradiction $p(\bar{w})$ is not a critical point of J, i.e., $\|(d_1, d_2)\| \neq 0$. We have three cases: (I) $d_1 \neq 0, d_2 = 0$, (II) $d_1 = 0, d_2 \neq 0$, (III) $d_1 \neq 0, d_2 \neq 0$. Since cases (I), (II) are symmetric to each other, essentially we only need to prove cases (I) and (III).

Case (I). Observe that $w^{k_i} \to \bar{w}$ implies $d^{k_i} \to \nabla J(p(\bar{w})) = (d_1, d_2)$ and (d_1, d_2) is the only limit point of d^{k_i}. Since $\|d^{k_i}\| \neq 0$ for all k_i (due to Alg. III.1) and $\|(d_1, d_2)\| \neq 0$, there exists $\delta > 0$ such that

$$\max\left\{\|d_1^{k_i}\|, \|d_2^{k_i}\|\right\} > \delta,\ \forall k_i. \tag{3.10}$$

This together with the fact $\|d_1^{k_i}\| \to \|d_1\| > 0, \|d_2^{k_i}\| \to \|d_2\| = 0$ as $k_i \to \infty$ leads to

$$\|d_1^{k_i}\| = \max\left\{\|d_1^{k_i}\|, \|d_2^{k_i}\|\right\} > \delta,\ \text{when } k_i \text{ is large.} \tag{3.11}$$

For each k_i with $\|d_1^{k_i}\| \neq 0$, by Lemma III.3 and Prop. III.1, there exists $\bar{s}_0 > 0$ such that

$$J(p(w^{k_i,2})) - J(p(w^{k_i,1})) \geq \left|J(p(w^{k_i})) - J(p(w^{k_i,1}))\right| > \tfrac{|t_1^{k_i}|}{8}s_1||d_1^{k_i}||^2 > 0, \tag{3.12}$$

$\forall 0 < s_1 \leq \bar{s}_0$. Observing $|t_1^{k_i}| \to |\bar{t}_1| > 0$ when $k_i \to \infty$ leads to $|t_1^{k_i}| \geq |\bar{t}_1|/2 > 0$, for k_i sufficiently large.

On the other hand, by conditions (i) and (ii), we have $\mathrm{dist}(p_1(\bar{w}), L_1) = |\bar{t}_1 \bar{w}_1| \geq \alpha > 0$ or $|\bar{t}_1| \geq \frac{\alpha}{\|\bar{w}_1\|} > 0$. Hence for k_i sufficiently large, we have

$$|t_1^{k_i}| \geq |\bar{t}_1|/2 \geq \frac{\alpha}{2\|\bar{w}_1\|} > 0, \tag{3.13}$$

which together with (3.11) and (3.12) yields

$$J(p(w^{k_i,2})) - J(p(w^{k_i,1})) > \tfrac{|t_1^{k_i}|}{8} s_1 ||d_1^{k_i}||^2 \geq \tfrac{1}{8} \tfrac{\alpha}{2\|\bar{w}_1\|} s_1 \delta^2 > 0. \tag{3.14}$$

The left hand side of the above inequality going to 0 under condition (iii) implies that $\lim\limits_{k_i \to \infty} s_1 \to 0$. This violates the uniform stepsize rule as stated in Theorem III.2. Therefore, $p(\bar{w})$ is a critical point of J.

Case (III). We write $\rho = \min\{\|d_1\|, \|d_2\|\}$ and $\rho^{k_i} = \min\{\|d_1^{k_i}\|, \|d_2^{k_i}\|\}$. Clearly, $\rho > 0$ because $d_1 \neq 0, d_2 \neq 0$. Observe that $\rho^{k_i} \to \rho$ since $w^{k_i} \to \bar{w}$, $J \in C^1$ and $p, \|\cdot\|$ are continuous functions. It then follows that there are at most finite many k_i's with $\rho^{k_i} < \rho/2$ and hence, without loss of generality, we can assume that $\rho^{k_i} \geq \rho/2, \forall k_i$.

For each k_i, again by Lemma III.3 and Prop. III.1, there exists $\bar{s}_0 > 0$ such that

$$J(p(w^{k_i,2})) - J(p(w^{k_i,1})) \geq \Big| J(p(w^{k_i})) - J(p(w^{k_i,1})) \Big| > \tfrac{|t_1^{k_i}|}{8} s_1 ||d_1^{k_i}||^2 > 0, \tag{3.15}$$

$\forall 0 < s_1 \leq \bar{s}_0$.

Combining (3.13) and (3.15) yields

$$J(p(w^{k_i,2})) - J(p(w^{k_i,1})) > \tfrac{|t_1^{k_i}|}{8} s_1 ||d_1^{k_i}||^2 \geq \tfrac{|t_1^{k_i}|}{8} s_1 (\rho^{k_i})^2 \geq \tfrac{1}{8} \tfrac{\alpha s_1}{2\|\bar{w}_1\|} (\tfrac{\rho}{2})^2 > 0 \tag{3.16}$$

for k_i sufficiently large.

Once again, letting $k_i \to \infty$ will lead to a contradiction with the uniform stepsize rule established in Theorem III.2. Therefore, in this case $p(\bar{w})$ must also be a critical point of J. The proof is complete. ∎

The preceding theorem shows that any convergent subsequence of w^k yields a co-existing critical point of J. However, such a convergent subsequence is not guaranteed without some stronger assumptions on J and on the sequences $\{w^k\}, \{w^{k,1}\}, \{w^{k,2}\}$ produced by Alg. III.1. Before we can prove our next theorem on the convergence of LMMOA, we need an auxiliary lemma to estimate the distance $\|w^{k+1} - w^k\|$ by using the two byproducts $w^{k,1}$ and $w^{k,2}$.

Define a function $j : H \longrightarrow \{1, 2\}$ by

$$j(d) = \begin{cases} 1, & \text{if } \|d_1\| \geq \|d_2\|, \\ 2, & \text{otherwise,} \end{cases} \tag{3.17}$$

for each $d = (d_1, d_2) \in H$, then we have the following lemma.

Lemma III.4 *For any* $k\,(\geq 1)$ *such that* $\max\{\|d_1^k\|, \|d_2^k\|\} > \epsilon$, *there hold*

$$\|w^{k+1} - w^k\| < 2\sqrt{2}\|w^{k,2} - w^{k,1}\| \tag{3.18}$$

and

$$\|w^{k+1} - w^k\| < (\sqrt{2}T + 1)\|w^{k,j(d^k)} - w^k\|, \tag{3.19}$$

where $d^k \equiv (d_1^k, d_2^k) = \nabla J(w^k)$, $0 < \epsilon < 1$ *and* $T(\geq 1)$ *are the parameters given in Alg. III.1 and* $j(\cdot)$ *is the functional defined in (3.17).*

Proof. By the stepsize rule in Alg. III.1, the condition $\max\{\|d_1^k\|, \|d_2^k\|\} > \epsilon$ implies that at least one of the two stepsizes $\bar{s}_1, \bar{s}_2$ is not equal to zero. Since p is an L-$\perp$

selection, i.e., $w_i^k \perp d_i^k$ $(i = 1, 2)$, with (3.7) we have

$$\begin{aligned}
\|w^{k+1} - w^{k,2}\|^2 &= \|\phi(\bar{s}_1, \bar{s}_2) - \phi(0, \bar{s}_2)\|^2 \\
&= \left\| \tfrac{(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|} - \tfrac{(w_1^k, w_2^k(\bar{s}_2))}{\|(w_1^k, w_2^k(\bar{s}_2))\|} \right\|^2 \\
&= \left\| \left(\tfrac{1}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|} - \tfrac{1}{\|(w_1^k, w_2^k(\bar{s}_2))\|} \right) (w_1^k, w_2^k(\bar{s}_2)) \right\|^2 + \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} \\
&= \left(\tfrac{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\| - \|(w_1^k, w_2^k(\bar{s}_2))\|}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|} \right)^2 + \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} \\
&\leq \left(\tfrac{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2)) - (w_1^k, w_2^k(\bar{s}_2))\|}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|} \right)^2 + \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} \\
&= \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} + \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} = \tfrac{2\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} \\
&\leq \tfrac{2\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k)\|^2}.
\end{aligned} \tag{3.20}$$

Similarly, we can obtain

$$\|w^{k,2} - w^k\|^2 \leq \frac{2\bar{s}_2^2 \|d_2^k\|^2}{\|(w_1^k, w_2^k(\bar{s}_2))\|^2}. \tag{3.21}$$

On the other hand, using $w_i^k \perp d_i^k$ $(i = 1, 2)$ and $\|w^k\| = \|(w_1^k, w_2^k)\| = 1$ gives

$$\begin{aligned}
\|w^{k,2} - w^{k,1}\|^2 &= \|\phi(0, \bar{s}_2) - \phi(\bar{s}_1, 0)\|^2 \\
&= \left(\tfrac{1}{\|(w_1^k, w_2^k(\bar{s}_2))\|} - \tfrac{1}{\|(w_1^k(\bar{s}_1), w_2^k)\|} \right)^2 + \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k)\|^2} + \tfrac{\bar{s}_2^2 \|d_2^k\|^2}{\|(w_1^k, w_2^k(\bar{s}_2))\|^2} \\
&\geq \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k)\|^2} + \tfrac{\bar{s}_2^2 \|d_2^k\|^2}{\|(w_1^k, w_2^k(\bar{s}_2))\|^2}, \\
&\quad \text{(the above inequality is strict if } \|d_1^k\| \leq \epsilon \text{ or } \|d_2^k\| \leq \epsilon)
\end{aligned} \tag{3.22}$$

where the last summation is strictly greater than $\frac{\bar{s}_1^2 \|d_1^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k)\|^2}$ and $\frac{\bar{s}_2^2 \|d_2^k\|^2}{\|(w_1^k, w_2^k(\bar{s}_2))\|^2}$ if $\min\{\|d_1^k\|, \|d_2^k\|\} > \epsilon$. This together with (3.20)-(3.21) and $\max\{\|d_1^k\|, \|d_2^k\|\} > \epsilon$ leads to

$$\|w^{k+1} - w^{k,2}\|^2 < 2\|w^{k,2} - w^{k,1}\|^2, \quad \|w^{k,2} - w^k\|^2 < 2\|w^{k,2} - w^{k,1}\|^2, \tag{3.23}$$

or

$$\|w^{k+1} - w^{k,2}\| < \sqrt{2}\|w^{k,2} - w^{k,1}\|, \quad \|w^{k,2} - w^k\| < \sqrt{2}\|w^{k,2} - w^{k,1}\|. \tag{3.24}$$

Hence by the triangle inequality, it follows that

$$\|w^{k+1} - w^k\| \le (\|w^{k+1} - w^{k,2}\| + \|w^{k,2} - w^k\|) < 2\sqrt{2}\|w^{k,2} - w^{k,1}\|. \tag{3.25}$$

Next, to prove (3.19), without loss of generality, suppose $j(d^k) = 2$, i.e., $\|d_1^k\| < \|d_2^k\|$, from which the stepsize rule in Alg. III.1 gives $\bar{s}_2 > 0$ and $\bar{s}_1 \le T\bar{s}_2$. This together with (3.20) leads to

$$\begin{aligned}
\|w^{k,2} - w^k\|^2 &= \|\phi(0, \bar{s}_2) - w^k\|^2 \\
&= \left\| \tfrac{(w_1^k, w_2^k(\bar{s}_2))}{\|(w_1^k, w_2^k(\bar{s}_2))\|} - w^k \right\|^2 \\
&= \left\| \tfrac{w^k + (0, sign(r_0^k)\bar{s}_2 d_2^k)}{\|(w_1^k, w_2^k(\bar{s}_2))\|} - w^k \right\|^2 \\
&= \left\| \left(\tfrac{1}{\|(w_1^k, w_2^k(\bar{s}_2))\|} - 1 \right) \right\|^2 + \tfrac{\bar{s}_2^2 \|d_2^k\|^2}{\|(w_1^k, w_2^k(\bar{s}_2))\|^2} \quad (\text{since } \|w^k\| = 1) \\
&> \tfrac{\bar{s}_2^2 \|d_2^k\|^2}{\|(w_1^k, w_2^k(\bar{s}_2))\|^2} \quad (\text{since } \|d_2^k\| = \max\{\|d_1^k\|, \|d_2^k\|\} > \epsilon > 0) \\
&\ge \tfrac{\bar{s}_2^2 \|d_2^k\|^2}{\|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} \quad (\text{since } w_1^k \perp d_1^k) \\
&\ge \tfrac{\bar{s}_1^2 \|d_1^k\|^2}{T^2 \|(w_1^k(\bar{s}_1), w_2^k(\bar{s}_2))\|^2} \quad (\text{since } \bar{s}_1 \le T\bar{s}_2, \|d_1^k\| < \|d_2^k\|) \\
&\ge \tfrac{1}{2T^2} \|w^{k+1} - w^{k,2}\|^2 (\text{due to the 2nd last inequality in (3.20)})
\end{aligned} \tag{3.26}$$

or equivalently

$$\|w^{k+1} - w^{k,2}\| < \sqrt{2}T\|w^{k,2} - w^k\|. \tag{3.27}$$

Finally, using the triangle inequality again gives

$$\|w^{k+1} - w^k\| \le (\|w^{k+1} - w^{k,2}\| + \|w^{k,2} - w^k\|) < (\sqrt{2}T + 1)\|w^{k,2} - w^k\|. \tag{3.28}$$

This proof is complete. ▮

Theorem III.4 *Let $J \in C^1(H_1 \times H_2, \mathbb{R})$ satisfy the (PS) condition, L_i be a closed subspace of H_i, $i = 1, 2$, $L = L_1 \times L_2$, $\{w^k\}, \{w^{k,1}\}, \{w^{k,2}\}$ be the sequences generated by Alg. III.1 when $\epsilon \to 0$. Assume further that the L-$\perp$ selection p satisfies*

(i) p is continuous,

(ii) *dist*$(p_i(w^k), L_i) \geq \alpha > 0$ $(i = 1, 2)$ *for some* $\alpha > 0$, $\forall k$,

(iii) $\displaystyle\sum_{k=1}^{\infty} \Big[J(p(w^{k,2})) - J(p(w^{k,1}))\Big] < \infty$ *and*

(iv) $\{J(p(w^k))\}$ *is bounded.*

Then the followings are true:

(a) *for any subsequence* $w^{k_i} \to \bar{w} \in S_{L^\perp}$, $p(\bar{w})$ *is a coexisting critical point of* J*;*

(b) $\{w^k\}$ *has a subsequence* $\{w^{k_i}\}$ *such that* $p(w^{k_i})$ *converges to a coexisting critical point of* J.

Proof. Only need to prove part (b), since part (a) is a direct consequence of Theorem III.3. To simplify the notations, we use the expression $j(k)$ for $j(d^k)$, $\forall k$, where $j(\cdot)$ is the functional defined in (3.17). Then $\|d^k_{j(k)}\| \equiv \max\Big\{\|d^k_1\|, \|d^k_2\|\Big\}$, $\forall k$.

First, we show that $\{w^k\}$ possesses a subsequence $\{w^{k_i}\}$ s.t. $\|\nabla J(p(w^{k_i}))\| \to 0$. Suppose not, by contradiction, there exists $\delta > 0$ such that

$$\|d^k_{j(k)}\| \equiv \max\Big\{\|d^k_1\|, \|d^k_2\|\Big\} > \delta, \ \forall k. \tag{3.29}$$

Applying Lemma III.3 and Prop. III.1, we have

$$\begin{aligned} J(p(w^{k,2})) - J(p(w^{k,1})) \ &\geq |J(p(w^{k,j(k)})) - J(p(w^k))| \\ &\geq \tfrac{|t^k_{j(k)}|}{4}\|d^k_{j(k)}\|\|w^{k,j(k)} - w^k\|, \ \forall k. \end{aligned} \tag{3.30}$$

Again, we write $p(w^k)$ as $p(w^k) = (t^k_1 w^k_1, t^k_2 w^k_2) + w^k_L$, where $t^k_1, t^k_2 \in \mathbb{R}$, $w^k_L \in L, \forall k$. Since $\text{dist}(p_i(w^k), L_i) = \|t^k_i w^k_i\| \geq \alpha > 0$ $(i = 1, 2)$ for every k, there exists $\rho > 0$ such that

$$|t^k_i| > \rho, \ i = 1, 2, \forall k,$$

from which, it follows that

$$|t^k_{j(k)}| > \rho, \ \forall k. \tag{3.31}$$

Taking (3.29) and (3.31) into account, (3.30) becomes

$$J(p(w^{k,2})) - J(p(w^{k,1})) \quad \geq \tfrac{|t^k_{j(k)}|}{4}\|d^k_{j(k)}\|\|w^{k,j(k)} - w^k\| > \tfrac{\rho\delta}{4}\|w^{k,j(k)} - w^k\|, \tag{3.32}$$

which together with Lemma III.4 yields

$$J(p(w^{k,2})) - J(p(w^{k,1})) > \tfrac{\rho\delta}{4}\|w^{k,j(k)} - w^k\| > \tfrac{\rho\delta}{4(\sqrt{2}T+1)}\|w^{k+1} - w^k\|. \tag{3.33}$$

With assumption (iii), we have

$$\sum_{k=1}^{\infty}\Big[\frac{\rho\delta}{4(\sqrt{2}T+1)}\|w^{k+1} - w^k\|\Big] < \sum_{k=1}^{\infty}\Big[J(p(w^{k,2})) - J(p(w^{k,1}))\Big] < \infty, \tag{3.34}$$

which implies that $\{w^k\}$ is a Cauchy sequence. Therefore $w^k \to \bar{w}$ for some $\bar{w} = (\bar{w}_1, \bar{w}_2) \in S_{L^\perp}$ with $\bar{w}_1 \neq 0, \bar{w}_2 \neq 0$. By continuity, $\max\{\|\bar{d}_1\|, \|\bar{d}_2\|\} \geq \delta > 0$, where $\bar{d} = (\bar{d}_1, \bar{d}_2) = \|\nabla J(p(\bar{w}))\|$. Now, we have three cases: (I) $\bar{d}_1 \neq 0, \bar{d}_2 = 0$, (II) $\bar{d}_1 = 0, \bar{d}_2 \neq 0$, (III) $\bar{d}_1 \neq 0, \bar{d}_2 \neq 0$. By the same lines in the proof of Theorem III.3, we can obtain a contradiction. Therefore, there exists a subsequence $\{w^{k_i}\}$ of $\{w^k\}$ such that $\|\nabla J(p(w^{k_i}))\| \to 0$ as $i \to \infty$. With assumptions (ii) and (iv), the (PS) condition shows that $\{p(w^{k_i})\}$ possesses a subsequence, still denoted by $\{p(w^{k_i})\}$, converging to a coexisting critical point of J. Thus (b) is proved. ∎

CHAPTER IV

INSTABILITY ANALYSIS ON SADDLE POINTS

In [67], instability analysis on saddle points was carried out in a single Hilbert space through a local minimax method. In this chapter we extend some instability analysis results therein via a local min-orthogonal method in two directions: from a local peak selection to a local L-$\perp$ selection and from a single Hilbert space (corresponding to a single equation case) to a product Hilbert space (corresponding to a system case). Based on a local min-orthogonal characterization, several estimates on the Morse index are established. Finally, a local instability index of a saddle point in a product Hilbert space is proposed and used to induce an order for multiple unstable solutions (saddle points) captured.

A. Motivation

Let H be a Hilbert space and $J : H \to \mathbb{R}$ be a C^1-functional. For a critical point u^* of J in H, assume $J''(u^*)$ is a self-adjoint Fredholm operator from $H \to H$. According to the spectral theory, H has an orthogonal spectral decomposition

$$H = H^- \oplus H^0 \oplus H^+$$

where H^-, H^0, H^+ are respectively the maximum negative definite, the null and the maximum positive definite subspaces of $J''(u^*)$ in H with $\dim(H^0) < \infty$, and are invariant under $J''(u^*)$. Then, the Morse index of u^* is $\mathrm{MI}(u^*) = \dim(H^-)$.

For a nondegenerate critical point u^* of J (implying $\dim(H^0) = 0$), the Morse index of u^* can be used to measure its local instability [55]; in other words, $\mathrm{MI}(u^*)$ can be used as a local instability index, a quantity measuring the maximum number of linearly independent directions along which a functional value can decrease, of J

at u^*. However, directly computing the Morse index of a critical point is usually very expensive since, essentially, one needs to find a basis for the unknown subspace H^-. Because of this, early results [4,5,6,14,56] focused on establishing some bound estimates of the Morse index based on a certain global minimax characterization. On the other hand, the Morse index is ineffective in measuring the local instability of a degenerate critical point since many situations can happen in the null space H^0. Recently, based on a local minimax characterization on saddle points, several bound estimates of the Morse index were established and a local minimax index (MMI) which is closely related to the Morse index was proposed to measure the local instability of saddle points not necessarily nondegenerate [40,67]. Unlike early results, these new estimates as well as MMI provide valuable guidance in finding saddle points with a prescribed Morse index. Also, MMI induces an order for unstable solutions, which is both useful and advantageous in numerical computations.

Let L be a closed subspace of H and $L \oplus L^\perp = H$ be its orthogonal decomposition. Denote by $S_{L^\perp} = \{u \in L^\perp : \|u\| = 1\}$ the unit sphere of $L^\perp$ and let $[L, v] = \{tv + v_L : t \in \mathbb{R}, v_L \in L\}$ for each $v \in S_{L^\perp}$. The following theorem is a result due to [67].

Theorem IV.1 ([67], Theorem 2.4) *Let $v^* \in S_{L^\perp}$. Assume that J has a local peak selection p w.r.t. L at v^*, $u^* \equiv p(v^*) \notin L$ and $v^* = \arg(loc)\min_{v \in S_{L^\perp}} J(p(v))$. If either $H^- \cap [L, v^*]^\perp = \{0\}$ or p is differential at v^*, then u^* is a critical point with*

$$\dim(L) + 1 = MI(u^*) + \dim(H^0 \cap [L, v^*]) \tag{4.1}$$

where H^0 is the null space of $J''(u^*)$ in H.

Due to the above theorem, a local minimax index of a saddle point u^* was proposed in [67] as $MMI(u^*) = \dim(L) + 1$. As noted in [67], MMI can measure the local instability for both degenerate and nondegenerate critical points. If $H^0 = \{0\}$, then $MMI(u^*) = MI(u^*)$. In this case, to capture a saddle point of $MI = n\,(\geq 1)$, a support L with $\dim(L) = n-1$ is needed in computation.

While instability analysis results established in [67] were based on a local minimax characterization (method), analogous results via a local min-orthogonal characterization (method) have not been investigated yet; in particular, the relation between the Morse index and a product support $\prod_{i=1}^m L_i\,(m \geq 2)$ is unknown. Besides, a local min-orthogonal method was used in [16] to find multiple coexisting saddle points in a certain order; meanwhile, further analysis especially analysis of instability was not fulfilled therein. These lie in our motivation on instability analysis.

B. Solutions in a Single Hilbert Space

For a closed subspace L of H, we still use same notations with same meanings, i.e., $S_{L^\perp}$, $[L, v]$ ($\forall v \in S_{L^\perp}$), as in the previous section. Assume $J \in C^1(H, \mathbb{R})$ and denote by ∇J its gradient.

Definition IV.1 ([66]) *A set-valued mapping $P : S_{L^\perp} \to 2^H$ is called an L-$\perp$ mapping of J if*

$$P(v) = \Big\{u \in [L, v] : \nabla J(u) \perp [L, v]\Big\}, \forall v \in S_{L^\perp}.$$

A single-valued mapping $p : S_{L^\perp} \to H$ is called an L-$\perp$ selection of J if $p(v) \in P(v)$, $\forall v \in S_{L^\perp}$. For a given $v \in S_{L^\perp}$, if p is locally defined on $\mathcal{N}(v) \cap S_{L^\perp}$ where $\mathcal{N}(v)$ is a neighborhood of v, then such p is called a local L-$\perp$ selection of J at v; in addition, if $p(w)$ is a local maximum of J in $[L, w]$ for each $w \in \mathcal{N}(v) \cap S_{L^\perp}$, then such p is called a local peak selection of J w.r.t. L at v.

In the rest of this chapter we always assume that $J''(u^*)$ is a self-adjoint, Fredholm operator from $H \to H$ whenever u^* is a critical point of J. Hence the orthogonal spectral decomposition of H is always available, i.e., $H = H^- \oplus H^0 \oplus H^+$, where H^-, H^0, H^+ are, respectively, the maximum negative definite, the null and the maximum positive definite subspaces of $J''(u^*)$ in H.

The following lemma gives stronger results than that of Lemma 2.3 in [67].

Lemma IV.1 *Let $v^* \in S_{L^\perp}$, $\mathbb{P}$ be the projection operator from H onto $[L, v^*]^\perp$. Suppose there exists a locally defined mapping $p : \mathcal{N}(v^*) \cap S_{L^\perp} \to H$ on some open neighborhood $\mathcal{N}(v^*)$ of v^* s.t. $p(v) \in [L, v]$ for any $v \in \mathcal{N}(v^*) \cap S_{L^\perp}$ and, in particular, $p(v^*) = t_0 v^* + v_L^*$ for some $v_L^* \in L$. If p is differentiable at v^* and $t_0 \neq 0$, then*

(i) $\mathbb{P}(p'(v^*)(w)) = t_0 w, \forall w \in [L, v^*]^\perp$;

(ii) $\mathbb{P}(p'(v^*)([L, v^*]^\perp)) = [L, v^*]^\perp$;

(iii) $p'(v^*)([L, v^*]^\perp) \oplus [L, v^*] = H$.

Proof. (i) Let $n = \dim(L)$ and $l_1, ..., l_n$ be a basis of L. For any $0 \neq w \in [L, v^*]^\perp$, denote $\tilde{v}(s) = \frac{v^* + sw}{\|v^* + sw\|}$. Clearly, $\tilde{v}(s)$ is a differentiable function of s with $\tilde{v}(s)|_{s=0} = v^*, \frac{d\tilde{v}(s)}{ds}|_{s=0} = w$. Since p is differentiable at v^*, then $p(\tilde{v}(s))$ is differentiable at $s = 0$. More precisely, there exists $s_0 > 0$ s.t. $p(\tilde{v}(s)) = t(s)\tilde{v}(s) + \sum_{i=1}^n \alpha_i(s) l_i$ for some differentiable (at 0) functions $t, \alpha_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, ..., n)$ with $t(0) = t_0$ and $\sum_{i=1}^n \alpha_i(0) l_i = v_L^*$, when $|s| < s_0$. Next,

$$\begin{aligned} p'(v^*)(w) &= \frac{dp(\tilde{v}(s))}{ds}\Big|_{s=0} \\ &= \Big[t'(s)\tilde{v}(s) + t(s)\tilde{v}'(s) + \textstyle\sum_{i=1}^n \alpha_i'(s) l_i\Big]\Big|_{s=0} \\ &= t'(0)v^* + t_0 w + \textstyle\sum_{i=1}^n \alpha_i'(0) l_i. \end{aligned}$$

Thus, $\mathbb{P}(p'(v^*)(w)) = t_0 w$. Finally, (i) leads to (ii) and (i)-(ii) lead to (iii). ∎

Lemma IV.2 *Let $v^* \in S_{L^\perp}$. Assume that J has a local L-$\perp$ selection p w.r.t. L at v^* s.t. p is differentiable at v^*, where $v^* = arg(loc) \min_{v \in S_{L^\perp}} J(p(v))$. If $u^* = p(v^*) \notin L$,* *then* u^* *is a critical point of* J *with*

$$p'(v^*)([L, v^*]^\perp) \cap H^- = \{0\} \tag{4.2}$$

and

$$MI(u^*) \leq \dim(L) + 1. \tag{4.3}$$

Proof. Suppose (4.2) does not hold, then there is $w \in [L, v^*]^\perp$ with $||w|| = 1$ s.t.

$$p'(v^*)(w) \in H^- \cap p'(v^*)([L, v^*]^\perp).$$

Close to $u^* = p(v^*)$, we have the second order Taylor expansion

$$J(u) = J(u^*) + \frac{1}{2}\langle J''(u^*)(u - u^*), u - u^*\rangle + o(||u - u^*||^2). \tag{4.4}$$

Letting $v^*(s) = \frac{v^*+sw}{||v^*+sw||}$, we have $v^*(s) \in \mathcal{N}(v^*) \cap S_{L^\perp}$ for $|s|$ is small and $\frac{dv^*(s)}{ds}|_{s=0} = w$. It then follows that

$$u(s) = p(v^*(s)) = u^* + sp'(v^*)(w) + o(|s|). \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$J(p(v^*(s))) = J(u^*) + \frac{s^2}{2}\langle J''(u^*)(p'(v^*)(w)), p'(v^*)(w)\rangle + o(s^2) < J(u^*),$$

where the last strict inequality holds true for $|s|$ sufficiently small since

$$p'(v^*)(w) \in H^- \quad \textit{implies} \quad \langle J''(u^*)(p'(v^*)(w)), (p'(v^*)(w))\rangle < 0.$$

The above inequality violates the assumption that v^* is a local minimum of $J \circ p$ on $S_{L^\perp}$. Thus, (4.2) is proved.

To prove (4.3), suppose by contradiction $MI(u^*) = \dim(H^-) > \dim(L) + 1$. By Lemma IV.1, we have $H = p'(v^*)([L, v^*]^\perp) \oplus [L, v^*]$. Applying the decomposition

$H = H^- \oplus H^0 \oplus H^+$ leads to $p'(v^*)([L, v^*]^\perp) \cap H^- \neq \{0\}$. This violates (4.2). ∎

Next, for a given symmetric matrix $Q \in \mathbb{R}^{n\times n}$, we denote respectively the positive eigenspace, the negative eigenspace and the kernel of Q in $\mathbb{R}^n$ by $Q^+, Q^-, ker(Q)$. Clearly, $\mathbb{R}^n = Q^- \oplus Q^+ \oplus ker(Q)$.

Lemma IV.3 *Let $L = span\{u_1, u_2, \ldots, u_n\}$, where $u_i's \in H$ are linearly independent. Suppose $v^* \in S_{L^\perp}$ and p is a local L-⊥ selection of J w.r.t. L at v^* such that (a) p is continuous at v^*, (b) $u^* = p(v^*) \notin L$, (c) $v^* = arg(loc)min_{v\in S_{L^\perp}} J(p(v))$. Let*

$$Q = \begin{pmatrix} \langle J''(u^*)v^*, v^*\rangle & \langle J''(u^*)u_1, v^*\rangle & \cdots & \langle J''(u^*)u_n, v^*\rangle \\ \langle J''(u^*)v^*, u_1\rangle & \langle J''(u^*)u_1, u_1\rangle & \cdots & \langle J''(u^*)u_n, u_1\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle J''(u^*)v^*, u_n\rangle & \langle J''(u^*)u_1, u_n\rangle & \cdots & \langle J''(u^*)u_n, u_n\rangle \end{pmatrix} \tag{4.6}$$

and define

$$G^+ = \{t_0v^* + t_1u_1 + \ldots + t_nu_n : (t_0, t_1, \ldots, t_n)^T \in Q^+\} \subseteq [L, v^*], \tag{4.7}$$

$$G^- = \{t_0v^* + t_1u_1 + \ldots + t_nu_n : (t_0, t_1, \ldots, t_n)^T \in Q^-\} \subseteq [L, v^*], \tag{4.8}$$

$$G^0 = \{t_0v^* + t_1u_1 + \ldots + t_nu_n : (t_0, t_1, \ldots, t_n)^T \in ker(Q)\} \subseteq [L, v^*], \tag{4.9}$$

then the followings hold:

(i) u^* *is a critical point of* J*;*

(ii) $[L, v^*] = G^- \oplus G^0 \oplus G^+$*;*

(iii) $\dim(H^0 \cap [L, v^*]) \leq \dim(G^0) = \dim(ker(Q))$*;*

(iv) $\dim(L) + 1 - \dim(Q^+) - \dim(H^0 \cap [L, v^*]) \leq MI(u^*)$*;*

(v) $\dim(L) + 1 - \dim(Q^+) - \dim(ker(Q)) \leq MI(u^*)$.

Proof. (i) See Lemma 2.5 in [66].

(ii) By the definitions of G^+, G^-, G^0, we have

$$\dim(Q^+) = \dim(G^+), \dim(Q^-) = \dim(G^-), \dim(ker(Q)) = \dim(G^0). \tag{4.10}$$

Then

$$\begin{aligned} n+1 &= \dim(Q^-) + \dim(ker(Q)) + \dim(Q^+) \\ &= \dim(G^-) + \dim(G^0) + \dim(G^+). \end{aligned} \tag{4.11}$$

Observe that $G^+ \cap G^- = G^+ \cap G^0 = G^- \cap G^0 = \{0\}$, then

$$\begin{aligned} \dim(G^- + G^0 + G^+) &= \dim(G^-) + \dim(G^0) + \dim(G^+) \\ &= n+1 = \dim([L, v^*]). \end{aligned} \tag{4.12}$$

Since $G^- + G^0 + G^+ \subseteq [L, v^*]$, we have $[L, v^*] = G^- \oplus G^0 \oplus G^+$.

(iii) Directly from the definition of G^0.

(iv) Write H as $H = H^- \oplus (H^0 \cap [L, v^*]) \oplus (H^0 \cap [L, v^*])^{\perp}_{H^0} \oplus H^+$. By (ii), $H = [L, v^*] \oplus [L, v^*]^{\perp} = G^- \oplus G^0 \oplus G^+ \oplus [L, v^*]^{\perp}$. If $\dim(L) + 1 - \dim(Q^+) > MI(u^*) + \dim(H^0 \cap [L, v^*])$, i.e.,

$$\dim(G^- \oplus G^0) = \dim(L) + 1 - \dim(G^+) > MI(u^*) + \dim(H^0 \cap [L, v^*]),$$

then

$$(G^- \oplus G^0) \cap ((H^0 \cap [L, v^*])^{\perp}_{H^0} \oplus H^+) \neq \{0\}.$$

For any $w \in (G^- \oplus G^0) \cap ((H^0 \cap [L, v^*])^{\perp}_{H^0} \oplus H^+)$ with $w = t_0 v^* + t_1 u_1 + \ldots + t_n u_n \neq 0$ and some $(t_0, t_1, \ldots, t_n)^T \in Q^- \oplus ker(Q)$, we can write it as $w = w^0 + w^+$, where $w^0 \in (H^0 \cap [L, v^*])^{\perp}_{H^0}$, $w^+ \in H^+$. We claim that $w^+ \neq 0$. If $w^+ = 0$, then $w = w^0 \in (G^- \oplus G^0) \cap (H^0 \cap [L, v^*])^{\perp}_{H^0}$, where

$$\begin{aligned} (G^- \oplus G^0) \cap (H^0 \cap [L, v^*])^{\perp}_{H^0} &\subseteq [L, v^*] \cap (H^0 \cap [L, v^*])^{\perp}_{H^0} \\ &= (H^0 \cap [L, v^*]) \cap (H^0 \cap [L, v^*])^{\perp}_{H^0} = \{0\}. \end{aligned}$$

A contradiction. Thus $w^+ \neq 0$. Then the inequality

$$\langle J''(u^*)w, w\rangle = \langle J''(u^*)(w^0 + w^+), (w^0 + w^+)\rangle = \langle J''(u^*)w^+, w^+\rangle \ > 0$$

contradicts the assumption that

$$\langle J''(u^*)w, w\rangle = (t_0, t_1, \dots, t_n)Q(t_0, t_1, \dots, t_n)^T \leq 0,$$

where $(t_0, t_1, \dots, t_n)^T \in Q^- \oplus ker(Q)$. Thus,

$$\dim(L) + 1 - \dim(Q^+) - \dim(H^0 \cap [L, v^*]) \leq MI(u^*).$$

(v) Combining (iii) and (iv) yields (v). ∎

Theorem IV.2 *Under the assumptions in Lemma IV.3, if $H^- \cap [L, v^*]^\perp = \{0\}$ or p is differentiable at v^*, then*

$$\dim(L) + 1 - \dim(Q^+) - \dim(ker(Q)) \leq MI(u^*) \leq \dim(L) + 1.$$

Proof. By Lemma IV.3(v), it only needs to prove the right inequality in the above. If p is differentiable at v^*, it clearly holds true by Lemma IV.2. Next, assume $H^- \cap [L, v^*]^\perp = \{0\}$. Suppose the right inequality does not hold, i.e., $MI(u^*) > \dim(L)+1$. Then with the orthogonal decomposition $H = H^- \oplus H^0 \oplus H^+ = [L, v^*] \oplus [L, v^*]^\perp$, we have $H^- \cap [L, v^*]^\perp \neq \{0\}$. A contradiction. Thus, $MI(u^*) \leq \dim(L) + 1$. ∎

Note that the condition $H^- \cap [L, v^*]^\perp = \{0\}$ posed in the above theorem can not be verified numerically since H^- is usually unknown and $[L, v^*]^\perp$ is infinite-dimensional. However, if the Hessian matrix Q is nonsingular, i.e., $|Q| \neq 0$, then the local L-$\perp$ selection p becomes differentiable at v^*. In this case, a better result can be obtained as in the next theorem.

Theorem IV.3 *Under the assumptions in Lemma IV.3, if* $|Q| \neq 0$*, i.e.,* $ker(Q) = \{0\}$*, then the followings hold:*

(i) p *is differentiable at* v^* *and* u^* *is a critical point of* J*;*

(ii) $\dim(G^0) = \dim(ker(Q)) = 0$ *or* $[L, v^*] = G^+ \oplus G^-$*;*

(iii) $\dim(H^0 \cap [L, v^*]) = 0$*;*

(iv) $\dim(L) + 1 - \dim(Q^+) \leq MI(u^*) \leq \dim(L) + 1$.

Proof. (i) By the Implicit Function Theorem, $|Q| \neq 0$ implies that p is differentiable at v^*. Hence, u^* is a critical point of J, refer also to Lemma 2.5 in [66].

(ii) See part (ii) in Lemma IV.3 wherein $G^0 = \{0\}$.

(iii) See (iii) in Lemma IV.3.

(iv) Follows from Theorem IV.2 while applying (i) and (ii). ∎

Remark IV.1 Theorems IV.2 and IV.3 provide us a simple method for estimating the Morse index of a critical point u^* based on a local min-orthogonal characterization. In [67], the author gave a similar yet more accurate result on estimates of the Morse index based on a local minimax characterization. But, it involves the unknown invariant subspace H^0. Here, we only need to calculate $\dim(ker(Q))$ and $\dim(Q^+)$, a much more easier job to do numerically in comparison with computing $\dim(H^0 \cap [L, v^*])$. If Q is nonsingular, then a better estimate on MI can be achieved as in Theorem IV.3 (iv). Moreover, if p happens to be a local peak selection of J w.r.t. L at v^*, then $MI(u^*) = \dim(L) + 1$ since $\dim(Q^+) = 0$. This coincides with the results of Theorem 2.5 in [67].

The above analysis can be stated as below:

Corollary IV.1 *Under the assumptions in Lemma IV.3, if* $Q^+ = ker(Q) = \{0\}$ *and* $J \in C^2$ *in some neighborhood of* $p(v^*)$*, then* p *is a differentiable local peak selection of* J *w.r.t.* L *at* v^* *such that* $MI(p(v^*)) = \dim(L) + 1$*.*

C. Solutions in a Product Hilbert Space

In Definition IV.1, if replacing the space H with a product space $\prod_{i=1}^n H_i \, (n \geq 2)$, the support L with a product subspace $\prod_{i=1}^n L_i$, where L_i is a closed subspace of $H_i \, (i = 1, 2, \ldots, n)$, and denoting by $\nabla J = (\partial J_1, \partial J_2, \cdots, \partial J_n)$ the gradient of J, then we can extend the definition of an L-$\perp$ mapping (selection) as follows:

Definition IV.2 *A set-valued mapping* $P : S_{L^\perp} \to 2^H$ *is called an* L-$\perp$ *mapping of* J *if*

$$P(w) = \Big\{ u = (u_1, \cdots, u_n) \in \prod_{i=1}^n [L_i, w_i] : \partial J_j(u) \perp [L_j, w_j], j = 1, 2, \ldots, n \Big\},$$

$\forall w = (w_1, \cdots, w_n) \in S_{L^\perp}$*. A single-valued mapping* $p : S_{L^\perp} \to H$ *is called an* L-$\perp$ *selection of* J *if* $p(w) \in P(w)$, $\forall w \in S_{L^\perp}$*. For a given* $w \in S_{L^\perp}$*, if* p *is locally defined on* $\mathcal{N}(w) \cap S_{L^\perp}$ *where* $\mathcal{N}(w)$ *is a neighborhood of* w*, then such* p *is called a local* L-$\perp$ *selection of* J *at* w*; in addition, if* $p(\bar{w})$ *is a local maximum of* J *in* $\prod_{i=1}^n [L_i, \bar{w}_i]$ *for each* $\bar{w} = (\bar{w}_1, \cdots, \bar{w}_n) \in \mathcal{N}(w) \cap S_{L^\perp}$*, then such* p *is called a local peak selection of* J *w.r.t.* L *at* w*.*

Remark IV.2 Clearly, Definition IV.2 implies Definition IV.1 when $n \geq 2$, i.e., when H is a product space. To reduce the confusion between these two definitions, we consider Definition IV.1 as a special case ($n = 1$) of Definition IV.2. Moreover, when $n = 2$, Definition IV.2 becomes Definition II.1. One motivation to replace Definition IV.1 or II.1 with Definition IV.2 is to find the coexisting solutions to n-component systems. For example, several 3-component vector solitons have been found in Section V.A.2 by utilizing Definition IV.2.

Once estimates on the Morse index of critical points in a single Hilbert space are established, similar results for critical points in a product Hilbert space can be obtained via similar arguments. The results are based on our local min-orthogonal characterization established in Chapter II (see Theorem II.2). Next, we list two lemmas without proofs by extending the results of Lemma IV.1 and Lemma IV.2 to the case of a product Hilbert space, respectively. One can follow the same lines of the proofs to Lemmas IV.1 and IV.2.

Let $H = H_1 \times H_2$ be a product Hilbert space. For a critical point $u^* = (u_1^*, u_2^*)$ of a dual functional J on H, assume that $J''(u^*)$ is a self-adjoint, Fredholm operator from $H \to H$. Once again, we decompose H as $H = H^- \oplus H^0 \oplus H^+$, where H^-, H^0, H^+ are, respectively, the negative definite, the null and the positive definite (eigen) subspaces of $J''(u^*)$ in H.

Lemma IV.4 *Let $w^* = (w_1^*, w_2^*) \in S_{L_1^\perp \times L_2^\perp}$ with $w_1^* \neq 0, w_2^* \neq 0$, $\mathbb{P}$ be the projection operator from $H_1 \times H_2$ onto $[L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp$ and p be a local L-$\perp$ selection of J w.r.t. $L_1 \times L_2$ at w^* s.t. $p(w^*) \equiv (p_1(w^*), p_2(w^*)) = (t_1 w_1^* + w_{L_1}^*, t_2 w_2^* + w_{L_2}^*)$ for some $(w_{L_1}^*, w_{L_2}^*) \in L_1 \times L_2$. If $t_1 t_2 \neq 0$ and p is differentiable at w^*, then*

(i) $\mathbb{P}(p'(w^*)(\alpha)) = (t_1\alpha_1, t_2\alpha_2), \forall \alpha = (\alpha_1, \alpha_2) \in [L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp$*;*

(ii) $\mathbb{P}(p'(w^*)([L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp) = [L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp$*;*

(iii) $p'(w^*)([L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp) \oplus [L_1, w_1^*] \times [L_2, w_2^*] = H_1 \times H_2$*.*

Lemma IV.5 *Let $w^* = (w_1^*, w_2^*) \in S_{L_1^\perp \times L_2^\perp}$ with $w_1^* \neq 0$, $w_2^* \neq 0$ such that $w^* =$ arg(loc) $\min_{w \in S_{L_1^\perp \times L_2^\perp}} J(p(w))$, where $p(w^*) \equiv (p_1(w^*), p_2(w^*))$ is a local L-$\perp$ selection of J w.r.t. $L_1 \times L_2$ at w^* and differentiable at w^*. If $p_1(w^*) \notin L_1$ and $p_2(w^*) \notin L_2$,* *then* $p(w^*)$ *is a critical point of* J *with*

$$p'(w^*)([L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp) \cap H^- = \{0\} \tag{4.13}$$

and

$$MI(p(w^*)) \leq \dim(L) + 2 = \dim(L_1) + \dim(L_2) + 2. \tag{4.14}$$

With Lemmas IV.4 and IV.5, we are now ready to extend the results of Theorems IV.2-IV.3 to a product space as below (the proofs are similar to that of Theorems IV.2-IV.3 and thus omitted).

Theorem IV.4 *Let* $L_1 = span\{u_1, u_2, \ldots, u_m\}$, $L_2 = span\{v_1, v_2, \ldots, v_n\}$, *where* u_i *are orthogonal vectors* ($i = 1, 2, \ldots, m$) *and so are* v_j ($j = 1, 2, \ldots, n$). *Let* $w^* = (u_0, v_0) \in S_{L_1^\perp \times L_2^\perp}$ *with* $u_0 \neq 0$, $v_0 \neq 0$ *s.t.*

$$w^* = arg(loc) \min_{w \in S_{L_1^\perp \times L_2^\perp}} J(p(w))$$

where $p(w) = (\sum_{i=0}^m t_i u_i, \sum_{j=0}^n s_j v_j)$ *is a local L-*$\perp$ *selection of* J *w.r.t.* $L_1 \times L_2$ *and differentiable at* w^*. *Denote* $p(w^*) = (\sum_{i=0}^m \bar{t}_i u_i, \sum_{j=0}^n \bar{s}_j v_j)$ *and let* Q *be the Hessian matrix of the quadratic function* $\phi : \mathbb{R}^{m+n+2} \times \mathbb{R}^{m+n+2} \to \mathbb{R}$ *given by*

$$\phi(t_0, ..., t_m, s_0, ..., s_n) = \frac{1}{2}\langle J''(p(w^*))p(w), p(w)\rangle$$

at $(\bar{t}_0, ..., \bar{t}_m, \bar{s}_0 ..., \bar{s}_n)$. *If* $\bar{t}_0 \bar{s}_0 \neq 0$, *then* $p(w^*)$ *is a critical point of* J *with*

$$\begin{aligned} \dim(L_1 \times L_2) + 2 - \dim(ker(Q)) - \dim(Q^+) &\leq MI(p(w^*)) \\ &\leq \dim(L_1 \times L_2) + 2 \end{aligned} \tag{4.15}$$

where $Q^+ \subseteq \mathbb{R}^{m+n+2}$ *is the positive eigenspace of* Q. *Moreover, if* $Q^+ = ker(Q) = \{0\}$ *and* $J \in C^2$ *in some neighborhood of* $p(w^*)$, *then* p *is a differentiable local peak selection of* J *w.r.t.* L *at* w^* *such that* $MI(p(w^*)) = \dim(L_1) + \dim(L_2) + 2$.

Note that results in the above theorem can be easily generalized to an N-product space (corresponding to an N-component system). For example, under some similar assumptions as above, the Morse index of an N-component saddle point $p(w^*)$ is

$$MI(p(w^*)) = \sum_{i=1}^{N} \dim(L_i) + N.$$

D. Discussion on a Local Instability Index

In this section, we discuss how to formulate a properly useful local instability index (LII) for saddle points in a product Hilbert space. We first extend the results of Theorem IV.1 to a product Hilbert space. Then we propose a definition of LII so that it can be used as an partial order for multiple coexisting solutions.

When it comes to multiple solution problems, especially when the number of solutions is large or even infinite, a proper order of the solutions should be always advantageous both theoretically and numerically. An index used to define such order should make sense in the meaning that qualitative behaviors of the solutions, like instabilities, are properly measured or reflected so that two solutions assigned with different indices are comparable. Besides, the order designated should be very easy to find or calculate; otherwise, it may lose its advantages in numerical computations. It is known that the Morse index can be used to define such order for nondegenerate critical points. As suggested in Theorems IV.2-IV.3 and Corollary IV.1, see also [67], the number $\dim(L)$ naturally provides us with an order meeting the criterion just mentioned.

As before, we work in a 2-product Hilbert space $H = H_1 \times H_2$.

Lemma IV.6 *Let $w^* = (w_1^*, w_2^*) \in S_{L_1^\perp \times L_2^\perp}$ with $w_1^* \neq 0$, $w_2^* \neq 0$. Assume there exist a neighborhood $\mathcal{N}(w^*)$ of w^* and a locally defined mapping $p : \mathcal{N}(w^*) \cap S_{L_1^\perp \times L_2^\perp} \to H$*

s.t. $p(w) \in [L_1, w_1] \times [L_2, w_2]$ *for any* $w = (w_1, w_2) \in \mathcal{N}(w^*) \cap S_{L_1^\perp \times L_2^\perp}$. *Denote*

$$H_0^- = H^- \oplus (H^0 \cap [L_1, w_1^*] \times [L_2, w_2^*]) \tag{4.16}$$

and $p(w^*) = (p_1(w^*), p_2(w^*))$. *Assume further that* p *is differentiable at* w^* *and* $p_1(w^*) \notin L_1$ *and* $p_2(w^*) \notin L_2$. *If* $\dim(H_0^-) > \dim(L_1 \times L_2) + 2$, *then*

$$p'(w^*)([L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp) \cap (H_0^- \backslash (H^0 \cap [L_1, w_1^*] \times [L_2, w_2^*])) \neq \{0\} \tag{4.17}$$

but

$$p'(w^*)([L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp) \cap (H^0 \cap [L_1, w_1^*] \times [L_2, w_2^*]) = \{0\}. \tag{4.18}$$

Proof. The above lemma is a direct extension of Lemma 2.4 in [67], refer to its proof therein. ∎

With the above lemma, we can strengthen the results of Lemma IV.5 as follows:

Lemma IV.7 *Let* $w^* = (w_1^*, w_2^*) \in S_{L_1^\perp \times L_2^\perp}$ *with* $w_1^* \neq 0$, $w_2^* \neq 0$ *such that* $w^* = arg(loc) \min_{w \in S_{L_1^\perp \times L_2^\perp}} J(p(w))$, *where* $p(w^*) = (p_1(w^*), p_2(w^*))$ *is a local L-⊥ selection of* J *w.r.t.* $L_1 \times L_2$ *at* w^* *and differentiable at* w^*. *If* $p_1(w^*) \notin L_1$ *and* $p_2(w^*) \notin L_2$, *then* $p(w^*)$ *is a critical point of* J *with*

$$p'(w^*)([L_1, w_1^*]^\perp \times [L_2, w_2^*]^\perp) \cap H_0^- = \{0\} \tag{4.19}$$

and

$$\dim(H_0^-) = MI(p(w^*)) + \dim(H^0 \cap [L_1, w_1^*] \times [L_2, w_2^*]) \leq \dim(L) + 2. \tag{4.20}$$

Proof. Follows the same argument as in the proof of Lemma IV.2 while applying Lemma IV.6, refer also to the proof of Theorem 2.4 in [67]. ∎

When p is a local peak selection, then $\dim(H_0^-)$ has a lower bound as stated in the following lemma.

Lemma IV.8 *Let* $w^* = (w_1^*, w_2^*) \in S_{L_1^\perp \times L_2^\perp}$ *with* $w_1^* \neq 0$, $w_2^* \neq 0$. *Assume that* $p : \mathcal{N}(w^*) \cap S_{L_1^\perp \times L_2^\perp} \to H$ *is a local peak selection of* J *defined on* $\mathcal{N}(w^*) \cap S_{L_1^\perp \times L_2^\perp}$ *for some neighborhood* $\mathcal{N}(w^*)$ *of* w^*, *and denote* $p(w^*) = (p_1(w^*), p_2(w^*))$. *If* $p_1(w^*) \notin L_1$ *and* $p_2(w^*) \notin L_2$, *then*

$$\dim(L) + 2 \leq \dim(H_0^-) = MI(p(w^*)) + \dim(H^0 \cap [L_1, w_1^*] \times [L_2, w_2^*]). \qquad (4.21)$$

Proof. Follows the same steps of the proof of Theorem 2.2 in [67]. ∎

With Lemmas IV.7 and IV.8, we can extend the results of Theorem IV.1 to a product Hilbert space.

Theorem IV.5 *Let* $w^* = (w_1^*, w_2^*) \in S_{L_1^\perp \times L_2^\perp}$ *with* $w_1^* \neq 0$, $w_2^* \neq 0$ *such that* $w^* = \textit{arg(loc)} \min_{w \in S_{L_1^\perp \times L_2^\perp}} J(p(w))$, *where* $p(w^*) = (p_1(w^*), p_2(w^*))$ *is a local peak selection of* J *at* w^* *and differentiable at* w^*. *If* $p_1(w^*) \notin L_1$ *and* $p_2(w^*) \notin L_2$, *then* $p(w^*)$ *is a critical point of* J *with*

$$MI(p(w^*)) + \dim(H^0 \cap [L_1, w_1^*] \times [L_2, w_2^*]) = \dim(L) + 2. \qquad (4.22)$$

Proof. Combining Lemmas IV.7 and IV.8 yields (4.22). ∎

With Theorem IV.5, we are now ready to define a local instability index of critical points in a product space, a generalization of MMI proposed in [67].

Definition IV.3 *A subspace* $\widetilde{L}_1 \times \widetilde{L}_2 \subset H$ *is called a support of a critical point* $p(v^*)$ *of a dual functional* $J(\cdot, \cdot)$, *where* p *is a continuous local* L-$\perp$ *selection of* J *w.r.t.*

$\widetilde{L}_1 \times \widetilde{L}_2$ *at* v^* *if*

$$v^* \equiv (v_1^*, v_2^*) = \arg(loc) \min_{(v_1,v_2) \in S_{\widetilde{L}_1^\perp \times \widetilde{L}_2^\perp}} J(p(v_1, v_2)).$$

$\widetilde{L}_1 \times \widetilde{L}_2$ *is called a minimal support of* $p(v^*)$ *if there holds* $\dim(\widetilde{L}_1) \leq \dim(L_1)$, $\dim(\widetilde{L}_2) \leq \dim(L_2)$ *for any support* $L_1 \times L_2$ *of* $p(v^*)$.

Definition IV.4 *Let* $L_1 \times L_2$ *be a minimal support of a saddle point* $u^* = (u_1^*, u_2^*)$ *of a dual functional* $J(\cdot, \cdot)$ *on* H. *The local instability index of* J *at* u^* *is defined by*

$$LII(u^*) = \dim(L_1) + \dim(L_2) + \dim(span\{u_1^*\}) + \dim(span\{u_2^*\}).$$

Remark IV.3 (a) Generally, LII can measure the local instability of saddle points. For local minima, we simply set their LII by 0.

(b) We use $\dim(\text{span}\{u_1^*\}) + \dim(\text{span}\{u_2^*\})$ instead of the constant 2 so that Definition IV.4 can cover both coexisting and non-coexisting saddle points. Note that in addition to coexisting solutions (saddle points), there may also exist many non-coexisting ones in various applications. If one component of a saddle point is zero, then its corresponding LII should drop by 1 accordingly. For instance, suppose $(u_1^*, 0)$ is a saddle point of J with a zero support L, i.e., $\dim(L) = 0$, then $LII(u_1^*, 0) = \dim(\text{span}\{u_1^*\}) = 1$. This indeed coincides with MMI in the single space case.

(c) Definition IV.4 can be easily extended to an n-component saddle point problem. For example, for a 3-component saddle point $u^* = (u_1^*, u_2^*, u_3^*)$, its $LII(u^*) = \sum_{i=1}^{3}(\dim(L_i) + \dim(\text{span}\{u_i^*\}))$.

(d) LII provides a partial order for multiple unstable solutions and hence offers some guidance in computation. Some applications of LII will be illustrated in Section V.A.

CHAPTER V

NUMERICAL EXPERIMENTS

In this chapter, we carry out numerical experiments and present our numerical results to various model problems by applying the LMOM and LMMOM (as well as Alg. II.1 and Alg. III.1) developed in Chapters II-III. We start with an experiment in search for multiple 2-component vector solitons to (1.4) using Alg. II.1. Estimates of the Morse indices of those solitons are given. Several important properties (e.g., continuity or differentiability) of a particular L-$\perp$ selection function are verified. Then, by an extension version of Alg. II.1, we find several 3-component vector solitons as well. Next, we apply the LMMOM to solve two noncooperative elliptic systems: one with a positive nonlinear term (also referred to as noncooperative systems of definite type), another with a nonlinear term neither bounded from below nor from above (also referred to as noncooperative systems of indefinite type). Before presenting our numerical solutions, we also study and verify several important properties (e.g., existence, differentiability, separation) of an L-$\perp$ selection p as well as a solution manifold $\mathcal{M}$ it induces. Moreover, we show in Theorem V.3 that saddle points of certain type strongly indefinite functional are still saddle points even when the functional is restricted on $\mathcal{M}$. This hence implies that saddle points of infinite Morse index can be approximated by the LMMOM but not by the LMOM. Finally, for a special class of Hamiltonian elliptic systems, we apply the LMMOM again to solve two Hamiltonian system models (i.e., the Lane-Emden system and the nonlinear biharmonic problem) by employing their close relationship with noncooperative elliptic systems.

A. Cooperative Elliptic Systems

1. 2-Component Model

With LMOA (i.e., Alg. II.1) developed in Chapter II, we are in a position to carry out some numerical experiments for our model problem (1.6). But before that let us first prove some properties of problem (1.6) and of an L-$\perp$ selection p it induced.

Denote $H_1 = H_2 = H_0^1(\Omega)$ and $H = H_1 \times H_2$. For the function $G(u,v)$ in (1.7), we see that

$$\begin{cases} G_u(u,v) = -u + \frac{uI(u,v)}{1+\mu I(u,v)}, \\ G_v(u,v) = -\gamma v + \frac{vI(u,v)}{1+\mu I(u,v)}. \end{cases} \tag{5.1}$$

Hence, our model problem (1.6) is a special example of system (2.1). While coexisting solutions to (1.6) have been repeatedly observed in physics experiments [32,37], most existence results (see, e.g., [6,46,64,68]) in the literature focus on nonzero solutions not the coexisting ones.

Proposition V.1 *(a) Any solution (u,v) to (1.6) with $\gamma \in (0,1)$ satisfies $u \perp v$ in $L^2(\Omega)$ or $\int_\Omega uv dx = 0$. (b) If $\gamma \in (0,1]$, then (1.6) has a positive coexisting solution (u,v) only if $\gamma = 1$.*

Proof. Multiplying the first and the second equations in (1.6) by v and u, respectively, and integrating by parts yields

$$\begin{cases} \displaystyle\int_\Omega \Big[\Delta u \cdot v - uv + \frac{uvI(u,v)}{1+\mu I(u,v)}\Big] dx &= 0, \\ \displaystyle\int_\Omega \Big[\Delta u \cdot v - \gamma vu + \frac{vuI(u,v)}{1+\mu I(u,v)}\Big] dx &= 0. \end{cases} \tag{5.2}$$

Subtracting the second equation above from the first one leads to $(\gamma - 1)\int_\Omega uv dx = 0$. Then, (a) follows if $\gamma \in (0,1)$ and (b) follows if $\gamma \in (0,1]$ and $u > 0, v > 0$. ∎

Remark V.1 The orthogonality condition in Proposition V.1(a) gives us a hint on selecting initial guesses for the LMOA; more precisely, we try to choose initial guesses (u, v) such that $u \perp v$ in $L^2(\Omega)$ (refer also to the initial guesses used for computing those 2-component vector solitons at the end of this section). In addition, Proposition V.1(b) coincides with the experimental observations, "...because the state $|0, 0\rangle$, nodeless in both components, can exist only in the degenerate case $\gamma = 1 \ldots$"[45].

We have seen in Section II.B and Chapter IV that continuity and/or differentiability of a local L-$\perp$ selection p plays a significant role both in the local characterization on coexisting saddle points and in estimates of the Morse index. For problem (1.6) or equivalently its associated functional J in (2.2) with $G(u, v)$ satisfying (1.7), the following theorem gives us a judgment on the differentiability of such p.

Theorem V.1 *Let $L = \{0\} \times \{0\} \subset H$. For any $(\bar{u}, \bar{v}) \in S_{L^\perp}$ with $\bar{u}^2, \bar{v}^2$ linearly independent, assume that $p_0(\bar{u}, \bar{v}) = (\bar{t}_1\bar{u}, \bar{t}_2\bar{v})$ is an L-$\perp$ selection of J w.r.t. L at $(\bar{u}, \bar{v})$ such that $\bar{t}_1\bar{t}_2 \neq 0$. Then there exists a local peak selection p of J w.r.t. L s.t. p is differentiable at $(\bar{u}, \bar{v})$, $p(u, v) = (t_1u_1, t_2v)$ with $t_1t_2 \neq 0$ for every (u, v) in some neighborhood of $(\bar{u}, \bar{v})$ and $p(\bar{u}, \bar{v}) = p_0(\bar{u}, \bar{v})$.*

Proof. For convenience, denote $f(u, v) = G_u(u, v)$ and $g(u, v) = G_v(u, v)$. By Definition II.1, the L-$\perp$ selection function $p_0(\bar{u}, \bar{v}) = (\bar{t}_1\bar{u}, \bar{t}_2\bar{v})$ can be solved from

$$\begin{cases} F_1(\bar{u}, \bar{v}, t_1, t_2) \equiv \frac{\partial J}{\partial t_1}(t_1\bar{u}, t_2\bar{v}) = -\int_\Omega [\Delta(t_1\bar{u}) + f(t_1\bar{u}, t_2\bar{v})]\bar{u}dx = 0 \\ F_2(\bar{u}, \bar{v}, t_1, t_2) \equiv \frac{\partial J}{\partial t_2}(t_1\bar{u}, t_2\bar{v}) = -\int_\Omega [\Delta(t_2\bar{v}) + g(t_1\bar{u}, t_2\bar{v})]\bar{v}dx = 0, \end{cases} \tag{5.3}$$

or equivalently from

$$\begin{cases} -\int_\Omega \Delta\bar{u} \cdot \bar{u}dx = \frac{1}{t_1}\int_\Omega f(t_1\bar{u}, t_2\bar{v}) \cdot \bar{u}dx \\ -\int_\Omega \Delta\bar{v} \cdot \bar{v}dx = \frac{1}{t_2}\int_\Omega g(t_1\bar{u}, t_2\bar{v}) \cdot \bar{v}dx. \end{cases} \tag{5.4}$$

Then, one can obtain the associated Jacobian matrix

$$\begin{aligned}
Q &\equiv \frac{\partial(F_1,F_2)}{\partial(t_1,t_2)}\Big|_{(t_1,t_2)=(\bar{t}_1,\bar{t}_2)} = \begin{pmatrix} \frac{\partial^2 J}{\partial t_1^2}(t_1\bar{u},t_2\bar{v}) & \frac{\partial^2 J}{\partial t_1\partial t_2}(t_1\bar{u},t_2\bar{v}) \\ \frac{\partial^2 J}{\partial t_2\partial t_1}(t_1\bar{u},t_2\bar{v}) & \frac{\partial^2 J}{\partial t_2^2}(t_1\bar{u},t_2\bar{v}) \end{pmatrix}\Bigg|_{(t_1,t_2)=(\bar{t}_1,\bar{t}_2)} \\
&= \begin{pmatrix} \int_\Omega(-\Delta\bar{u} - f_u(\bar{t}_1\bar{u},\bar{t}_2\bar{v})\bar{u})\bar{u}dx & \int_\Omega -f_v(\bar{t}_1\bar{u},\bar{t}_2\bar{v})\bar{v}\bar{u}dx \\ \int_\Omega -g_u(\bar{t}_1\bar{u},\bar{t}_2\bar{v})\bar{u}\bar{v}dx & \int_\Omega(-\Delta\bar{v} - g_v(\bar{t}_1\bar{u},\bar{t}_2\bar{v})\bar{v})\bar{v}dx \end{pmatrix} \\
&\quad \text{(because } (\bar{t}_1,\bar{t}_2) \text{ solves (5.4))} \\
&= \begin{pmatrix} \frac{1}{\bar{t}_1^2}\int_\Omega(f - f_u\cdot(\bar{t}_1\bar{u}))(\bar{t}_1\bar{u})dx & \frac{1}{\bar{t}_1\bar{t}_2}\int_\Omega(-f_v\cdot(\bar{t}_2\bar{v}))(\bar{t}_1\bar{u})dx \\ \frac{1}{\bar{t}_1\bar{t}_2}\int_\Omega(-g_u\cdot(\bar{t}_1\bar{u}))(\bar{t}_2\bar{v})dx & \frac{1}{\bar{t}_2^2}\int_\Omega(g - g_v\cdot(\bar{t}_2\bar{v}))(\bar{t}_2\bar{v})dx \end{pmatrix} \\
&= \begin{pmatrix} \frac{-2}{(\bar{t}_1)^2}\int_\Omega \frac{(\bar{t}_1\bar{u})^4}{D^2}dx & \frac{-2}{\bar{t}_1\bar{t}_2}\int_\Omega \frac{(\bar{t}_1\bar{u})^2(\bar{t}_2\bar{v})^2}{D^2}dx \\ \frac{-2}{\bar{t}_1\bar{t}_2}\int_\Omega \frac{(\bar{t}_1\bar{u})^2(\bar{t}_2\bar{v})^2}{D^2}dx & \frac{-2}{(\bar{t}_2)^2}\int_\Omega \frac{(\bar{t}_2\bar{v})^4}{D^2}dx \end{pmatrix}
\end{aligned}$$

and its determinant

$$\begin{aligned}
|Q| &= \frac{4}{(\bar{t}_1\bar{t}_2)^2}\begin{vmatrix} \int_\Omega \frac{(\bar{t}_1\bar{u})^4}{D^2}dx & \int_\Omega \frac{(\bar{t}_1\bar{u})^2(\bar{t}_2\bar{v})^2}{D^2}dx \\ \int_\Omega \frac{(\bar{t}_1\bar{u})^2(\bar{t}_2\bar{v})^2}{D^2}dx & \int_\Omega \frac{(\bar{t}_2\bar{v})^4}{D^2}dx \end{vmatrix} \\
&= \frac{4}{(\bar{t}_1\bar{t}_2)^2}\left(\int_\Omega \frac{(\bar{t}_1\bar{u})^4}{D^2}dx\int_\Omega \frac{(\bar{t}_2\bar{v})^4}{D^2}dx - \left(\int_\Omega \frac{(\bar{t}_1\bar{u})^2(\bar{t}_2\bar{v})^2}{D^2}dx\right)^2\right),
\end{aligned}$$

where $D = 1 + \mu((\bar{t}_1\bar{u})^2 + (\bar{t}_2\bar{v})^2)$. Next, we show that $|Q| > 0$ ($\neq 0$). Observe that $\frac{(\bar{t}_1\bar{u})^2}{D}, \frac{(\bar{t}_2\bar{v})^2}{D} \in L^2(\Omega)$ since $|\frac{(\bar{t}_1\bar{u})^2}{D}| < \frac{1}{\mu}, |\frac{(\bar{t}_2\bar{v})^2}{D}| < \frac{1}{\mu}$. Then by the Cauchy-Schwarz inequality, it follows that

$$\left(\int_\Omega \frac{(\bar{t}_1\bar{u})^2(\bar{t}_2\bar{v})^2}{D^2}dx\right)^2 \leq \int_\Omega \frac{(\bar{t}_1\bar{u})^4}{D^2}dx\int_\Omega \frac{(\bar{t}_2\bar{v})^4}{D^2}dx \tag{5.5}$$

which becomes an equality if and only if $\frac{(\bar{t}_1\bar{u})^2}{D}$ and $\frac{(\bar{t}_2\bar{v})^2}{D}$ are linearly dependent in $L^2(\Omega)$ or equivalently $\bar{u}^2, \bar{v}^2$ are linearly dependent. But $\bar{u}^2, \bar{v}^2$ are linearly independent by assumption, thus inequality (5.5) is strict and hence $|Q| > 0$. By the Implicit Function Theorem, there exist an open neighborhood $\mathcal{N}(\bar{u},\bar{v})$ of $(\bar{u},\bar{v})$ and an open neighborhood $\mathcal{N}(\bar{t}_1,\bar{t}_2)$ of $(\bar{t}_1,\bar{t}_2)$ such that for every $(u,v) \in \mathcal{N}(\bar{u},\bar{v}) \cap S_{L^\perp}$, there exists a unique $(t_1, t_2) \in \mathcal{N}(\bar{t}_1, \bar{t}_2)$ solving system (5.3) (wherein $\bar{u}, \bar{v}$ are replaced by u, v, respectively). Moreover, $t_1(u, v), t_2(u, v)$ are differentiable functions of (u, v). Taking the condition $\bar{t}_1\bar{t}_2 \neq 0$ into account, a local L-$\perp$ selection function p given by $p(u, v) = (t_1 u, t_2 v)$ with $t_1 t_2 \neq 0$ is well-defined and differentiable for each $(u, v) \in \mathcal{N}(\bar{u}, \bar{v}) \cap S_{L^\perp}$. Finally, observe that the diagonal elements of Q are negative, then $|Q| > 0$ implies Q is strictly negative definite. From this and the differentiability of p, we conclude that p is indeed a local peak selection of J w.r.t. L at $(\bar{u}, \bar{v})$. ∎

Remark V.2 The linear independency of $\bar{u}^2, \bar{v}^2$ in Theorem V.1 implies the linear independency of $\bar{u}, \bar{v}$. Although we cannot derive general conditions to assure the differentiability of a local L-$\perp$ selection p for the case $L \neq \{0\} \times \{0\}$, we can always numerically check or track the invertibility of the Jacobian matrix Q which somehow provides some useful information about the differentiability of p. For example, if Q is invertible, then a local L-$\perp$ selection p is well-defined and differentiable; furthermore, if Q is strictly negative definite, then p becomes a differentiable local peak selection.

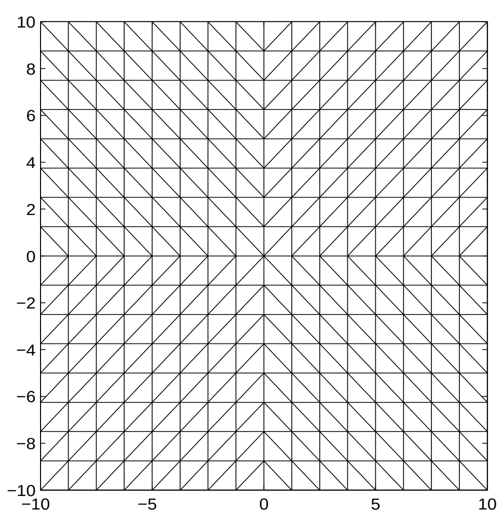

Fig. 3. A sample symmetric mesh on the square domain $\Omega = (-10, 10)^2$.

Next, we present our numerical solutions to our model problem (1.6). We choose $\gamma = 0.65, \mu = 0.5$, $\Omega = (-10, 10)^2$ (square), and use $\varepsilon = 10^{-4}$ as the tolerance for LMOA. Since the original physics model (1.4) as well as problem (1.6) possesses many symmetries, we develop symmetric mesh grids on Ω. Fig. 3 is a coarse version of a symmetric mesh that we used. The number of triangle elements used on Ω is 32,768.

Figs. 4-6 show the contour plots of the first few coexisting states (corresponding to dipole-, tripole-, quadrupole- and multipole-mode vector solitons) to (1.6) in an induced (partial) order based on their instability. In Fig. 4 (resp. Fig. 5 and Fig. 6), solutions (a)-(b) have the same local instability index as defined in Definition IV.4, so do solutions (c)-(d). Solutions (a) and (b) in Fig. 4 correspond respectively to the dipole-mode vector solitons with 45° orientation and 90° orientation; while solutions (a)-(b) in Fig. 5 correspond to two quadrupole-mode vector solitons with different orientations. Physically, dipole-mode vector solitons are the most stable vector solitons. On the other hand, based on our analysis in Chapter IV (see Theorem IV.4) as well as our numerical computations, dipole-mode vector solitons have the least Morse index (MI$\geq$ 3) among all the coexisting states. Hence, mathematically, dipole-mode solitons are still unstable, though “they are extremely robust objects.”[32]. Besides, it can be seen from Figs. 4-6 that for each solution (u, v), the 2nd component (i.e., v-component) always carries more complex solution structures than the 1st component (i.e., u-component) does. To be more precise, one sees that the v-component always has at least the same number of nodal lines as the u-component does. Take solution (a) or (b) in Fig. 6, for example, when u-component is nodal, then v-component must be nodal as well; conversely, v-component cannot be nodeless (i.e., bell-shaped) when u-component is nodal. In physics, u-component is called a fundamental mode (wave). It traps and guides a weaker higher-order mode (v-component) while traveling in a nonlinear bulk medium [26].

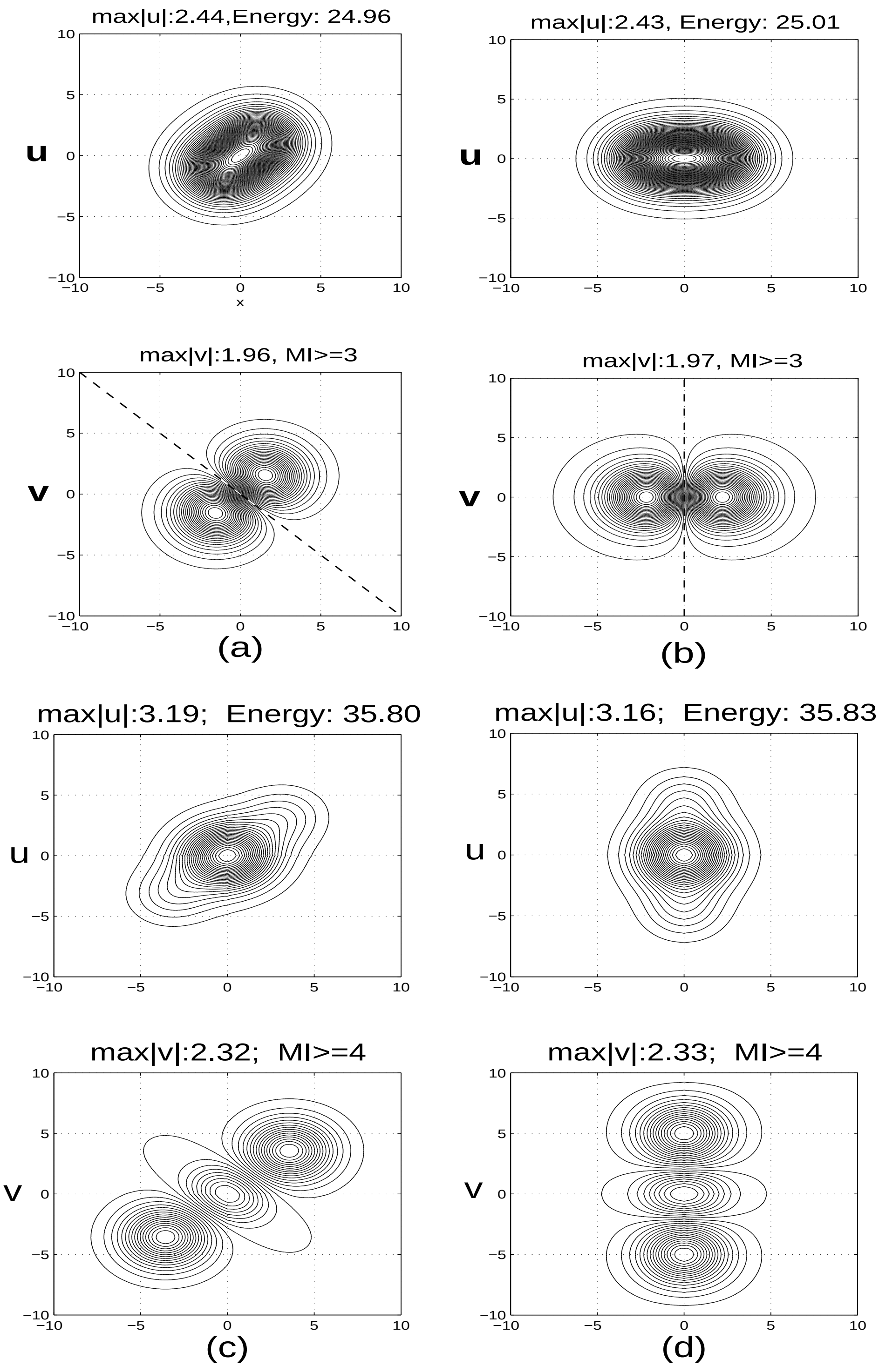

Fig. 4. Dipole- (top: a,b) and tripole-mode (bottom: c,d) vector solitons to (1.4) (i.e., solutions to (1.6) with $\Omega = (-10, 10)^2$, $\gamma = 0.65$, $\mu = 0.5$). Dashed lines indicate the nodal lines.

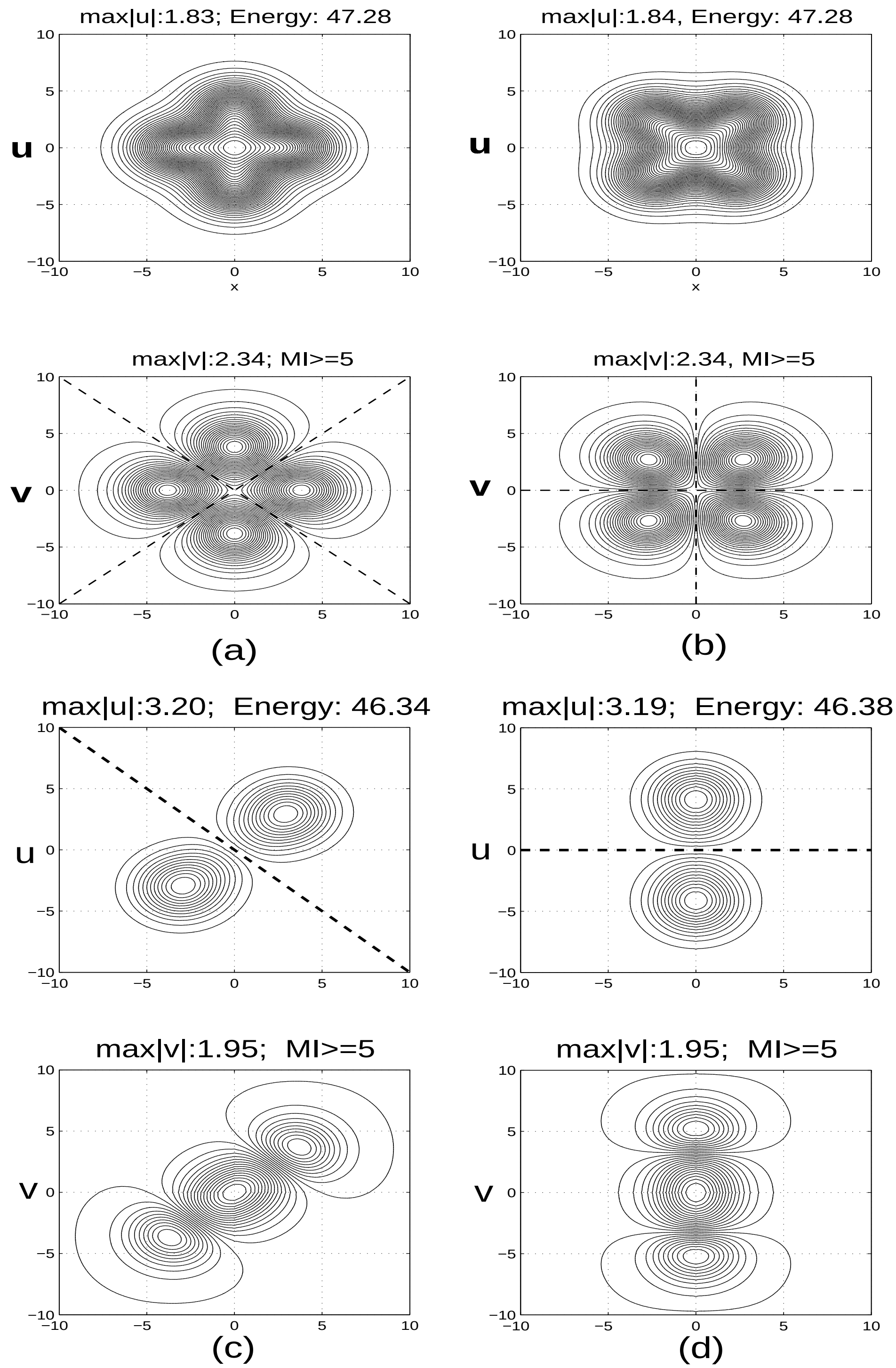

Fig. 5. Quadrupole- (top: a,b) and tripole-mode (bottom: c,d) vector solitons to (1.4) (i.e., solutions to (1.6) with $\Omega = (-10, 10)^2$, $\gamma = 0.65$, $\mu = 0.5$). Dashed lines indicate the nodal lines.

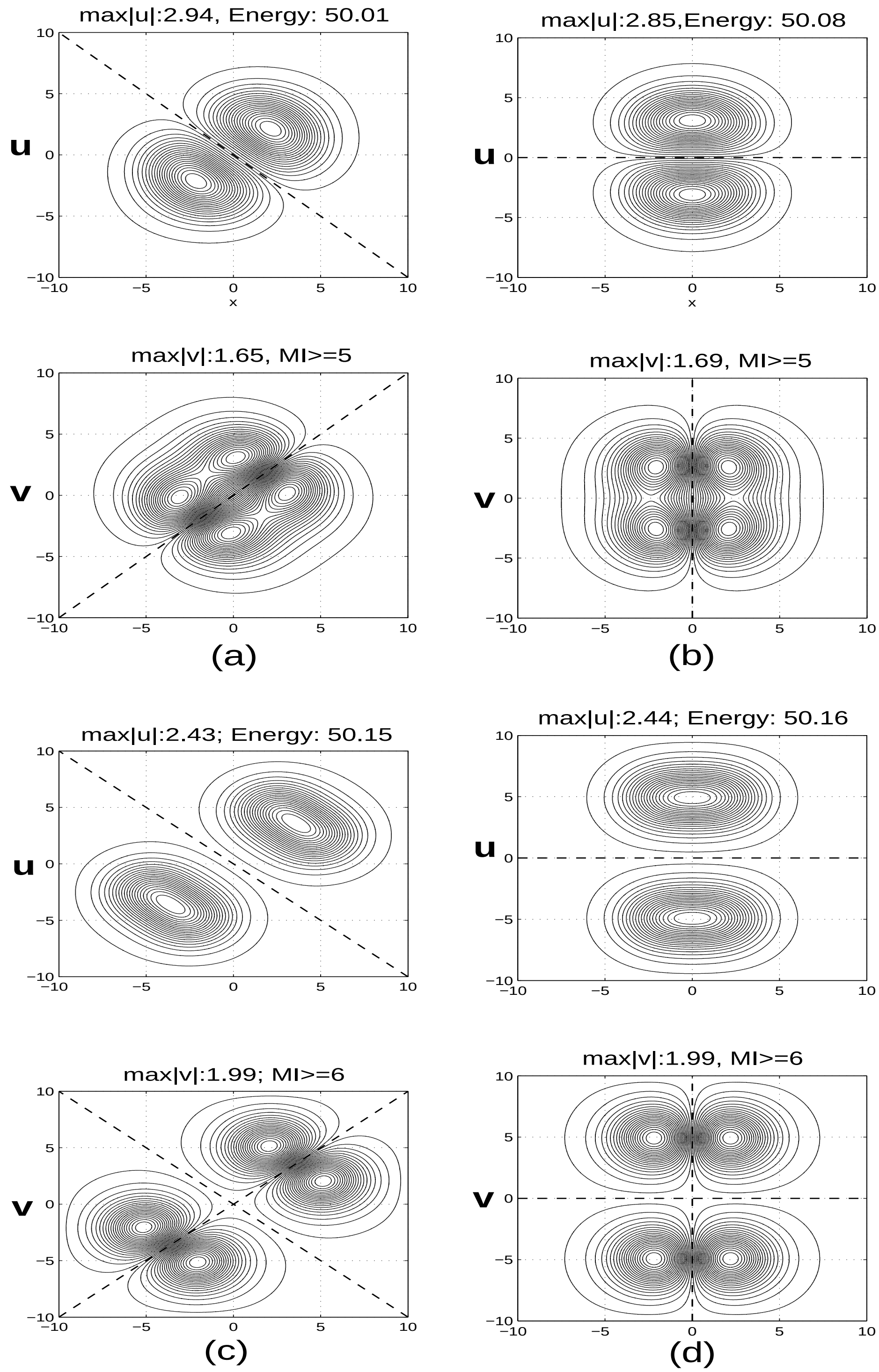

Fig. 6. Multipole-mode vector solitons to (1.4) (i.e., solutions to (1.6) with $\gamma = 0.65$, $\mu = 0.5$, $\Omega = (-10, 10)^2$). Dashed lines indicate the nodal lines.

The orthogonality condition stated in Proposition V.1(a) suggests us to construct the following supports L and initial guesses (u_0, v_0) to compute multiple coexisting states to problem (1.6), wherein

$$f(x,y) = e^{-0.05(x^2+y^2)}(y^2-100)(x^2-100),\ g(x,y) = \cos(0.05\pi x)\cos(0.05\pi y).$$

Clearly, $f(x,y), g(x,y)$ are positive symmetric functions in Ω and vanish on its boundary $\partial\Omega$. Note that the initial guesses listed below are just for the instructive purpose and readers' convenience. In fact, choosing initial guesses for the LMOA is quite flexible. For a large Morse index solution with certain symmetries, it is more advantageous and efficient to apply such symmetries as carried out in [60]. More precisely, we use the Haar projection (see, e.g., [17,60] and references therein) to preserve the symmetries, keeping or forcing our iterations in an invariant subspace induced by such symmetries. Even so a nonzero support sometimes is still required. Take the triple-mode vector solitons (refer to Fig. 5(c) or (d)), for example, we need a bell-shaped symmetric function (symmetric w.r.t. both the x and y axes) which serves as a sub-support for the v-component no matter what kind of symmetry we impose on that component. This is because for a triple-hump nodal function as in the second component of solution (c) or (d) in Fig. 5, without a sub-support, eventually it will collapse, i.e., it will decay to zero. Thus, as long as multiple solutions are concerned, introducing a support is essential in numerical computations.

(1) cf. Fig.4(a). Choose $(u_0, v_0) = (f(x,y), f(x,y)(y+x))$ and $L = \{0\} \times \{w_1\}$ with $w_1 = e^{-\frac{x^2+y^2}{50}} g(x,y)$. Solution (a) can also be obtained by applying the odd symmetry to the v-component w.r.t. the line $y + x = 0$ while letting $L = \{0\} \times \{0\}$. This is the most stable coexisting state that we can find. Its Morse index is at least 3.

(2) cf. Fig.4(b). Choose $(u_0, v_0) = (f(x,y), f(x,y)x)$ and use the same support L as in case (1). This solution can also be obtained by applying the odd symmetry to the v-component w.r.t. the y-axis while letting $L = \{0\} \times \{0\}$. Its Morse index is at least 3.

(3) cf. Fig.4(c). Choose $(u_0, v_0) = g(x,y)(e^{\frac{(x+y)^2}{50} - \frac{x^2+y^2}{20}}, e^{-\frac{(x+y)^2}{45}}(y+x+\frac{10}{3})(y+x-\frac{10}{3}))$ and $L = \{0\} \times \{w_1, w_2\}$ with $w_1 = e^{-\frac{(x+y)^2}{45}} g(x,y)$, $w_2 = (y+x)w_1$. This solution can also be found by applying the even symmetry w.r.t. the line $y + x = 0$ to the v-component while letting $L = \{0\} \times \{w_1\}$. Here, the sub-support $\{w_1\}$ for the v-component is essential; without it, the v-component will collapse (i.e., decay to zero), a situation that one needs to avoid in order to obtain coexisting solutions. Note that the v-component is a triple-hump sign changing function. The Morse index of this solution is at least 4.

(4) cf. Fig.4(d). Choose $u_0 = e^{\frac{y^2}{25} - \frac{x^2+y^2}{20}} g(x,y)$, $v_0 = e^{-\frac{(x+y)^2}{45}} g(x,y)(y + \frac{10}{3})(y - \frac{10}{3})$ and $L = \{0\} \times \{w_1, w_2\}$ with

$$w_1 = e^{-\frac{(x+y)^2}{45}} g(x,y), w_2 = e^{-\frac{(x+y)^2}{45}} \cos(0.05\pi x) \sin(0.1\pi y).$$

Similarly, solution (d) can be found by applying the even symmetry w.r.t. the x-axis to the v-component while letting $L = \{0\} \times \{w_1\}$. Again, the v-component is a triple-hump sign changing function for which the sub-support $\{w_1\}$ is necessary. The associated Morse index is at least 4.

(5) cf. Fig.5(a). Choose $(u_0, v_0) = f(x,y)(1, y^2 - x^2)$ and $L = \{0\} \times \{w_1, w_2, w_3\}$ with

$$w_1 = e^{-\frac{x^2+y^2}{50}} g(x,y), w_2 = e^{-\frac{x^2+y^2}{40}} g(x,y)(y-x), w_3 = e^{-\frac{x^2+y^2}{40}} g(x,y)(y+x).$$

Likewise, this solution can be obtained by letting $L = \{0\} \times \{0\}$ and applying the odd symmetry to the v-component w.r.t. both the line $y - x = 0$ and the line $y + x = 0$. This is a quadrupole-mode vector soliton with Morse index ≥ 5.

(6) cf. Fig.5(b). Choose $(u_0, v_0) = (e^{-\frac{x^2+y^2}{25}}(y^2 - 100)(x^2 - 100), f(x, y)xy)$ and $L = \{0\} \times \{w_1, w_2, w_3\}$ with

$$w_1 = e^{-\frac{x^2+y^2}{50}} g(x, y), w_2 = e^{-\frac{x^2+y^2}{40}} g(x, y)x, w_3 = e^{-\frac{x^2+y^2}{40}} g(x, y)y.$$

Likewise, this solution can be obtained by letting $L = \{0\} \times \{0\}$ and applying the odd symmetry to the v-component w.r.t. both the x and y axes. This is another quadrupole-mode vector soliton whose Morse index is at least 5.

(7) cf. Fig.5(c). Choose $(u_0, v_0) = (e^{-\frac{x^2+y^2}{10}}(y^2 - 100)(x^2 - 100)(y + x), f(x, y)xy)$ and apply the even (odd) symmetry w.r.t. the line $y + x = 0$ to the v- (u-) component, respectively. Meanwhile, a support $L = \{0\} \times \{w_1\}$ with $w_1 = e^{-\frac{x^2+y^2}{40}} g(x, y)$ is used. Again, the v-component is a triple-hump sign changing function for which the sub-support $\{w_1\}$ is required. The associated Morse index is at least 5.

(8) cf. Fig.5(d). Choose $u_0 = e^{-\frac{x^2+y^2}{15}} \sin(\frac{\pi y}{10}) \cos(\frac{\pi x}{20})$, $v_0 = e^{-\frac{x^2+y^2}{20}} g(x, y)(y^2 - x^2)$ and apply the even (odd) symmetry w.r.t. the x-axis to the v- (u-) component, respectively. Meanwhile, a support $L = \{0\} \times \{w_1\}$ with $w_1 = e^{-\frac{x^2+y^2}{50}} g(x, y)$ is used. Likewise, the v-component is a triple-hump sign changing function for which the sub-support $\{w_1\}$ is essential. The Morse index of this solution is at least 5.

(9) cf. Fig.6(a). Choose $(u_0, v_0) = (f(x,y)(y+x), f(x,y)(y-x)), L = \{0\} \times \{0\}$ and apply the odd symmetry w.r.t. the line $y + x = 0$ to the u-component and the odd (even) symmetry w.r.t. the line $y - x = 0$ ($y + x = 0$) to the v-component. This solution corresponds to a dipole-dipole mode vector soliton with Morse index at least 5.

(10) cf. Fig.6(b). Choose $(u_0, v_0) = (f(x,y)y, f(x,y)x), L = \{0\} \times \{0\}$ and apply the odd symmetry w.r.t. the x-axis to the u-component and the odd (even) symmetry w.r.t. the y-axis (x-axis) to the v-component. This is another dipole-dipole mode vector soliton with Morse index at least 5.

(11) cf. Fig.6(c). Choose $(u_0, v_0) = f(x,y)(y+x)(1, y-x)$ and $L = \{0\} \times \{0\}$. Then, apply the odd symmetry w.r.t. the line $y + x = 0$ to the u-component, and the odd symmetry w.r.t. both the lines $y + x = 0$ and $y - x = 0$ to the v-component, respectively. Its Morse index is at least 6.

(12) cf. Fig.6(d). Choose $(u_0, v_0) = f(x,y)y(1, x)$ and $L = \{0\} \times \{0\}$. Then, apply the odd symmetry w.r.t. the x-axis to the u-component, and the odd symmetry w.r.t. both the x and y axes to the v-component, respectively. Its Morse index is at least 6.

In addition to those coexisting solutions to (1.6), there are quite many non-coexisting ones which are of less interest in physics applications and hence are omitted here. With the LMOM developed in Chapter II, we are able to locate the coexisting solutions while excluding the non coexisting ones.

2. 3-Component Model

By extending the LMOM in Chapter II to the case of 3-component cooperative systems, we will find several 3-component spatial vector solitons in this subsection.

Consider an N-component vector soliton problem which describes the interaction of N mutually incoherent $(2+1)$-dimensional optical beams $E_1, ..., E_N$ propagating in a bulk saturable medium (e.g., photorefractive crystals) along z direction and can be modelled by an N-coupled NLS system [24,25,42]

$$i\frac{\partial E_j}{\partial z} + \Delta E_j - \frac{E_j}{1+\sum_{k=1}^{N}|E_k|^2} = 0, \; j = 1, 2, ..., N. \tag{5.6}$$

Here, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian. To describe and/or study soliton formation, soliton interactions and collisions, and so on, coexisting standing solitary wave solutions to (5.6) of the form

$$E_j = u_j(x, y)e^{-i\beta_j z}, \; j = 1, 2, ..., N, \tag{5.7}$$

are of particular interest in practice, where $0 < \beta_j < 1$ is the propagation constant and $u_j(x, y)$ is the envelope (i.e., amplitude) of the jth component. After introducing $\gamma_j = 1 - \beta_j$, $\lambda_j = \frac{\gamma_j}{\gamma_1}$ (note that $\lambda_1 \equiv 1$) and rescaling the amplitudes, $u_j \to \sqrt{\gamma_1}u_j$, and the coordinates, $\{x, y\} \to \{x/\sqrt{\gamma_1}, y/\sqrt{\gamma_1}\}$, we obtain an N-component semilinear elliptic system [24,25,42]

$$\Delta u_j(x, y) - \lambda_j u_j(x, y) + \tfrac{I}{1+\mu I}u_j(x, y) = 0, \; j = 1, 2, ..., N, \tag{5.8}$$

where $I = \sum_{j=1}^{N} u_j^2$ is the total intensity, $\mu \equiv \gamma_1 = 1 - \beta_1 \in (0, 1)$ is the effective saturation parameter (the case $\mu \to 0$ corresponds to the Kerr medium). Obviously, system (5.8) contains degeneracy, i.e., if one of u_j's is zero, then the dimension of the problem will drop by 1.

Next, we will find multiple coexisting solutions to (5.8) (corresponding to vector solitons to (5.6)) in an ascending order by their instability indices (i.e., their LIIs). We still use $\Omega = (-10, 10) \times (-10, 10) \subset \mathbb{R}^2$ (as was used in [32] and [62]) while imposing zero Dirichlet conditions on such Ω for system (5.8), and choose $\varepsilon = 10^{-4}$ as our tolerance to terminate our iterations. Again, the number of triangle elements used on Ω is 32,768.

Example V.1 *For system (5.8), let $N = 3$, then it becomes*

$$-\triangle u_j = G_{u_j}(u_1, u_2, u_3) = -\lambda_j u_j + \tfrac{u_1^2+u_2^2+u_3^2}{1+\mu(u_1^2+u_2^2+u_3^2)} u_j\,, \quad x \in \Omega, \quad j = 1, 2, 3 \tag{5.9}$$

where $\lambda_1 \equiv 1$, $G(u_1, u_2, u_3) = \frac{1}{2}\sum_{j=1}^{3}(\frac{1}{\mu} - \lambda_j)u_j^2 - \frac{\ln(1+\mu(u_1^2+u_2^2+u_3^2))}{2\mu^2}, u_1 = u_2 = u_3 = 0$ *on* $\partial\Omega$. *The associated energy functional is*

$$J(u_1, u_2, u_3) = \frac{1}{2}\int_\Omega (|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2)dx - \int_\Omega G(u_1, u_2, u_3)dx. \tag{5.10}$$

The following result is a direct generalization of Proposition V.1.

Proposition V.2 *Assume $\lambda_1 \neq \lambda_2 \neq \lambda_3$. For any solution (u_1, u_2, u_3) to (5.9), there holds $u_i \perp u_j$ in $L^2(\Omega)$ or $\int_\Omega u_i u_j dx = 0$ when $i \neq j$, $i, j = 1, 2, 3$.*

For problem (5.9), if $u_3 \equiv 0$, then it reduces to the 2-component vector soliton problem (1.6). In this sense, problem (1.6) is just a special case of Example V.1. The contour plots of the first few multiple coexisting unstable solutions to (5.9) with $\mu = 0.5, \lambda_2 = 0.8, \lambda_3 = 0.5$ (corresponding to 3-component vector solitons to (5.6) with $N = 3$) are shown in Figs. 7-9 and approximations of their Morse indices and LIIs are given in Table I, wherein $L = L_1 \times L_2 \times L_3 \subset (H_0^1(\Omega))^3$ are the supports used in our computations. We refer the reader to Remark IV.3(c) on the calculation of LIIs. Note that the actual Morse indices of some of those solutions may be different from the results in Table I. Even so Table I provides very good approximations on the Morse index according to both our numerical computations and related analysis carried out in Chapter IV.

Table I. Approximations on the Morse index (MI) and local instability index (LII) of 3-component vector solitons depicted in Figs. 7-9.

solution	Morse index	dim(L_1)	dim(L_2)	dim(L_3)	LII
Fig. 7: (a),(b)	5	0	1	1	5
Fig. 8: (a),(b)	6	0	1	2	6
Fig. 9: (a),(b)	7	0	1	3	7

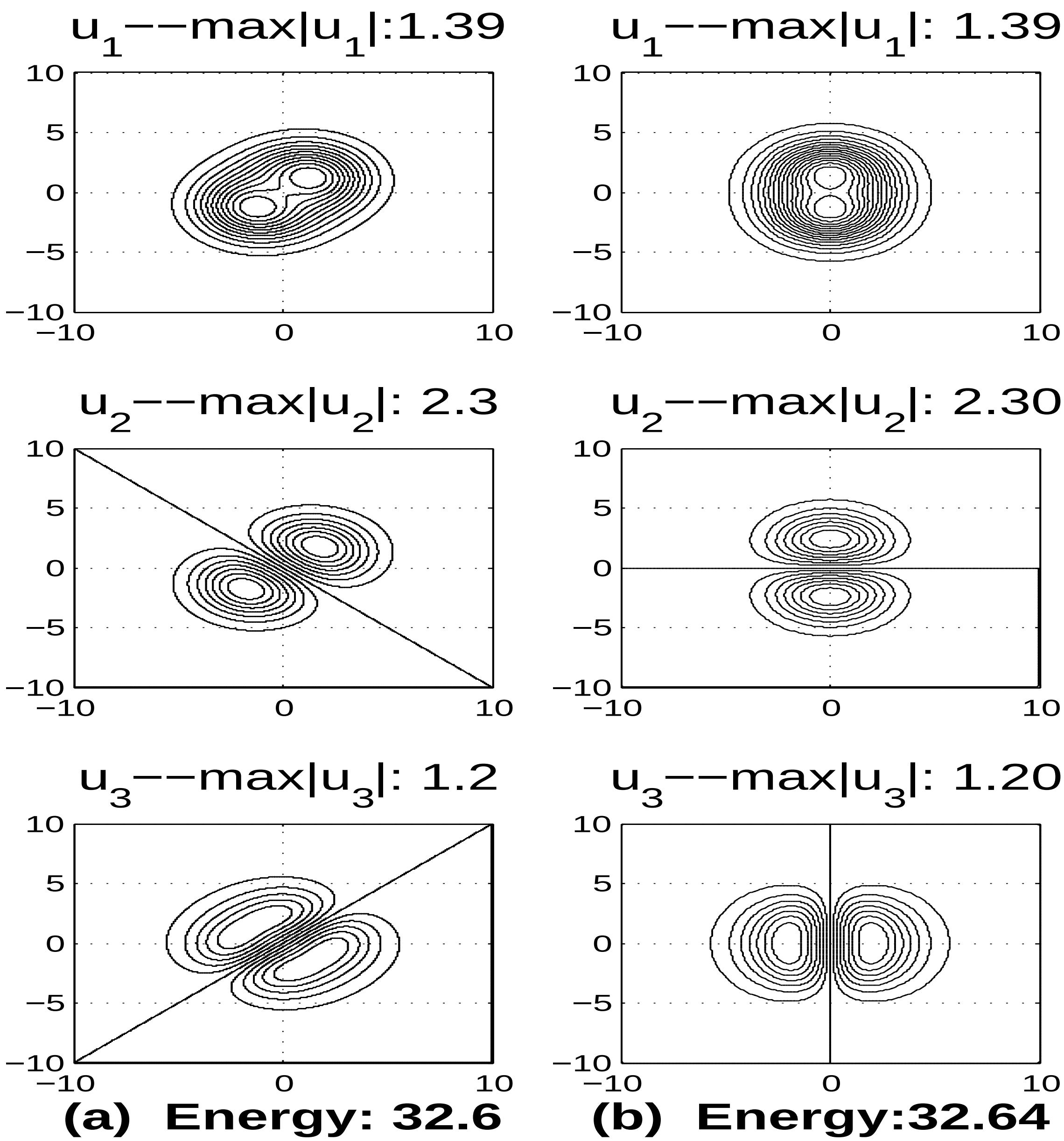

Fig. 7. 3-component dipole-dipole mode vector solitons to (5.6) with $N = 3$ (i.e., solutions to (5.9) with $\Omega = (-10, 10)^2$, $\mu = 0.5, \lambda_2 = 0.8, \lambda_3 = 0.5$). Solid lines indicate the nodal lines.

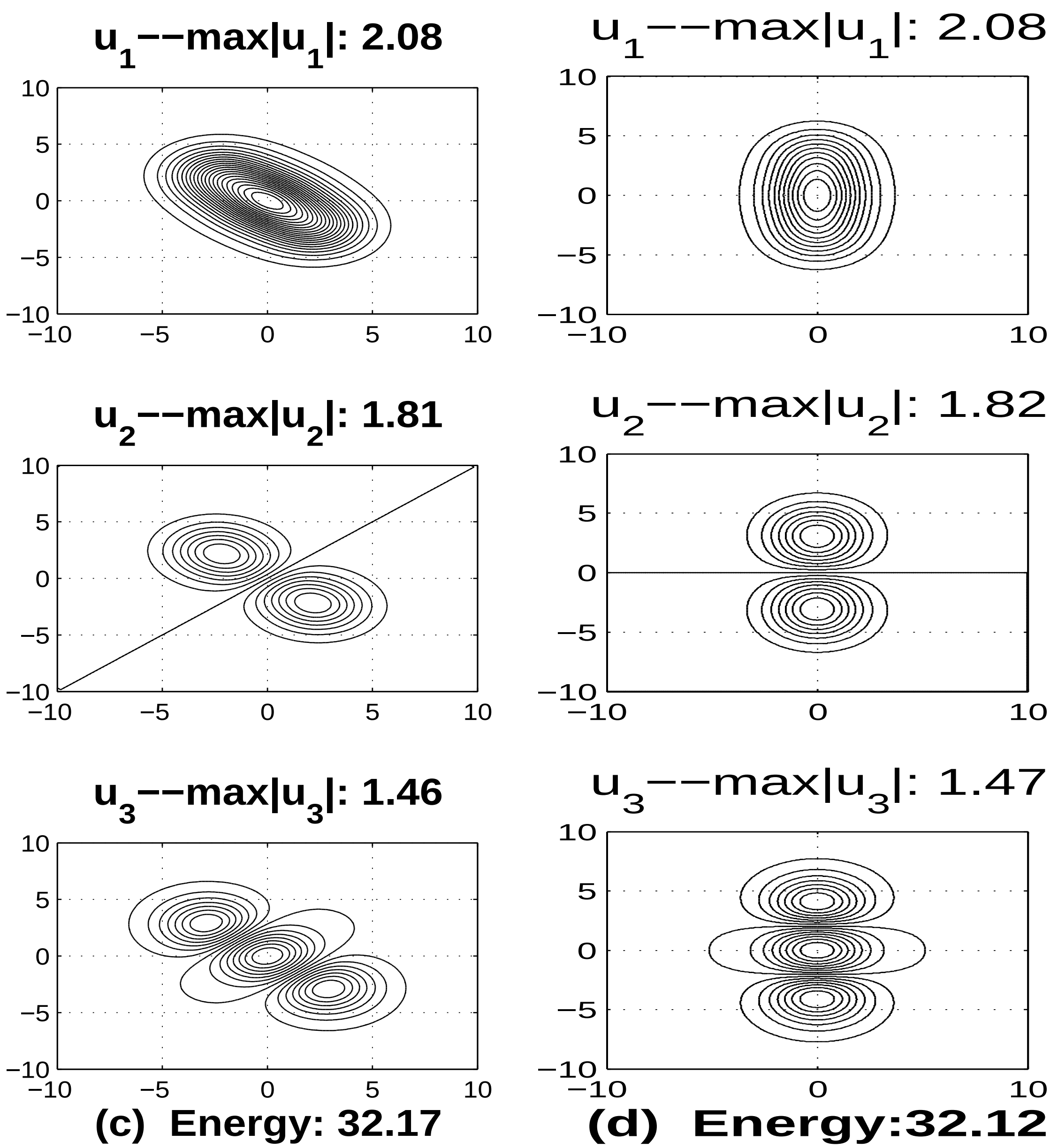

Fig. 8. 3-component dipole-tripole mode vector solitons to (5.6) with $N = 3$ (i.e., solutions to (5.9) with $\Omega = (-10, 10)^2$, $\mu = 0.5, \lambda_2 = 0.8, \lambda_3 = 0.5$). Solid lines indicate the nodal lines.

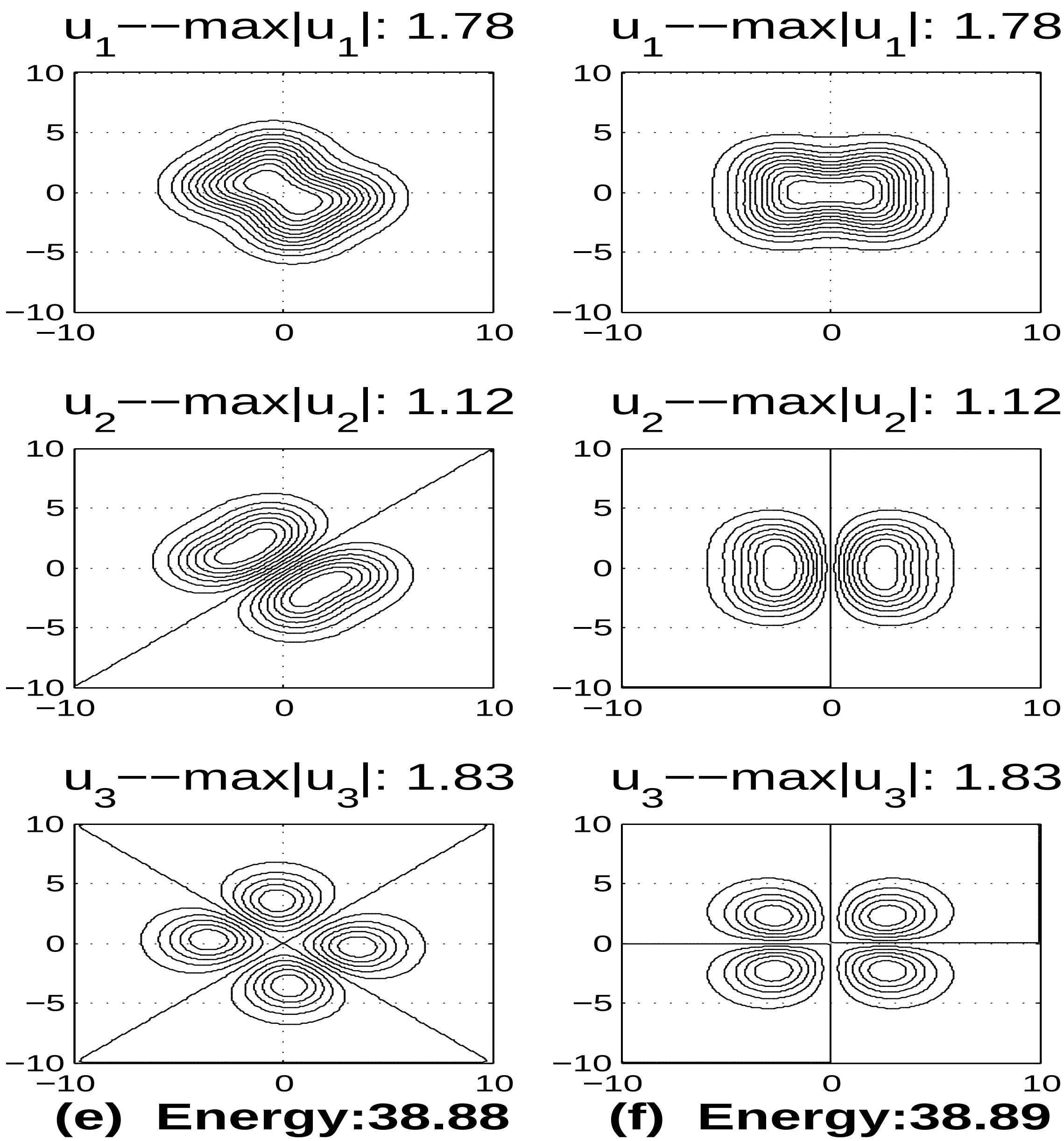

Fig. 9. 3-component dipole-quadrupole mode vector solitons to (5.6) with $N = 3$ (i.e., solutions to (5.9) with $\mu = 0.5, \lambda_2 = 0.8, \lambda_3 = 0.5$, $\Omega = (-10, 10)^2$). Solid lines indicate the nodal lines.

B. Noncooperative Elliptic Systems

In this section we apply the LMMOM developed in Chapter III to solve two noncooperative elliptic systems: one with a positive nonlinear term (called definite type), another with a nonlinear term neither bounded from below nor from above (called indefinite type). Before presenting our numerical solutions, we study several important properties such as existence, differentiability, and separation of an L-$\perp$ selection p and a solution manifold $\mathcal{M}$ it induces.

1. Definite Type

a. A Model

Consider noncooperative elliptic systems of the form [20,21,30,68]

$$\begin{cases} -\Delta u = \lambda u - \delta v + G_u(x; u, v) & x \in \Omega, \\ -\Delta v = \delta u + \gamma v - G_v(x; u, v) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \tag{5.11}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, $\gamma \leq \lambda, \delta > 0$. The nonlinear term $G(x; U) \in C^1(\Omega \times \mathbb{R}^2; \mathbb{R})$ (in the variables $U = (u, v) \in \mathbb{R}^2$) satisfies the following [20,21]

(F_1) $|\nabla G(x, U)| \leq c(1 + |U|^{\xi-1})$, $\quad \forall U \in \mathbb{R}^2$, a.e. $x \in \Omega$, for some $c > 0$ and $2 \leq \xi < \frac{2N}{N-2}$ if $N \geq 3$ or $2 \leq \xi < +\infty$ if $N = 1, 2$; (subcritical)

(F_2) $\displaystyle\liminf_{|U|\to\infty} \frac{U \cdot \nabla G(x; U) - 2G(x; U)}{|U|^\mu} \geq a > 0$ uniformly a.e. $x \in \Omega$ with $\mu > N(\xi - 2)/2$ if $N \geq 3$ or $\mu > \xi - 2$ if $N = 1, 2$; (nonquadratic)

(F_3) $G(x; U) \geq 0$, $\forall U \in \mathbb{R}^2$, $\displaystyle\lim_{|U|\to 0} \frac{G(x; U)}{|U|^2} = 0$ uniformly a.e. $x \in \Omega$.

If letting $H = L^2(\Omega) \times L^2(\Omega)$ and denoting by $\nabla G = (G_u, G_v)$ and

$$A = \begin{pmatrix} \lambda & -\delta \\ \delta & \gamma \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -\vec{\Delta} = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix},$$

then (5.11) becomes

$$\mathcal{L}U = \nabla G(x; U),$$

where $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ is a self-adjoint operator given by $\mathcal{L}U = R(-\vec{\Delta} - A)U$ with domain

$$D(\mathcal{L}) = W^{2,2}(\Omega, \mathbb{R}^2) \cap W_0^{1,2}(\Omega, \mathbb{R}^2).$$

Problem (5.11) was extensively investigated in [20,21], etc. As noted in [21], the following *asymptotic noncrossing* conditions

$$(F_4^+) \;\; \lambda_{k-1} < \liminf_{|U|\to\infty} \frac{2G(x;U)}{|U|^2} \le \limsup_{|U|\to\infty} \frac{2G(x;U)}{|U|^2} \le \lambda_k \quad \text{unif. a.e.} x \in \Omega$$

$$(F_4^-) \;\; \lambda_{k-1} \le \liminf_{|U|\to\infty} \frac{2G(x;U)}{|U|^2} \le \limsup_{|U|\to\infty} \frac{2G(x;U)}{|U|^2} < \lambda_k \quad \text{unif. a.e.} x \in \Omega$$

or *crossing* conditions

$$(F_5) \;\; G(x;U) \ge \frac{1}{2}\lambda_{k-1}|U|^2 \quad \text{a.e. } x \in \Omega, \; \forall U \in \mathbb{R}^2$$

$$(F_6) \;\; \limsup_{|U|\to 0} \frac{2G(x;U)}{|U|^2} \le \alpha < \lambda_k < \beta \le \liminf_{|U|\to\infty} \frac{2G(x;U)}{|U|^2} \;\; \text{unif. a.e.} x \in \Omega,$$

where $\lambda_{k-1} < \lambda_k$ are two consecutive eigenvalues of the operator $\mathcal{L}$, were used to assure the existence of nontrivial solutions to (5.11). In some sense the assumption $G(x;U) \ge 0, \forall U \in \mathbb{R}^2$ in (F_3) is a necessity for conditions $(F_4^{\pm})$ (*asymptotic noncrossing*) or (F_5)-(F_6) (*crossing*). Meanwhile, other authors [7,30,68] showed that such a condition may be weakened by, e.g., $G(x; 0, v) \ge 0$, for a.e. $x \in \Omega, v \in \mathbb{R}$, under which existence results can still be obtained.

Example V.2 *Choose $N = 2$ (i.e., $\Omega \subset \mathbb{R}^2$) and $G(x; u, v) \equiv G(u, v) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1}$ with $p, q > 1$. Then (5.11) becomes*

$$\begin{cases} -\Delta u = \lambda u - \delta v + |u|^{p-1}u & x \in \Omega, \\ -\Delta v = \delta u + \gamma v - |v|^{q-1}v & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega. \end{cases} \tag{5.12}$$

For this particular example, we can check that conditions (F_1)-(F_3) are satisfied. Let $H = H_0^1(\Omega) \times H_0^1(\Omega)$ and $\|\cdot\|$ be its usual norm, i.e., $\|(u, v)\|^2 = \int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx$, $\forall (u, v) \in H$. Then, weak solutions of (5.12) are critical points of the following C^2-functional on H

$$J(u, v) = \frac{1}{2}\int_\Omega \Big[(|\nabla u|^2 - |\nabla v|^2) - (\lambda u^2 - 2\delta uv - \gamma v^2)\Big]dx - \int_\Omega G(u, v)dx. \tag{5.13}$$

We have the following properties (i.e., Props. V.3-V.5) of J in (5.13).

Proposition V.3 *Every critical point of J or $-J$ has infinite Morse index.*

Proof. Only need to show that the conclusion holds true at the trivial critical point $(0, 0)$ of J. The second derivative of J at $(0, 0)$ is

$$J''(0, 0) = \begin{pmatrix} -\Delta - \lambda & \delta \\ \delta & \Delta + \gamma \end{pmatrix}.$$

For any kth eigenpair (λ_k, ϕ_k) of $(-\Delta, H_0^1(\Omega))$ with $\lambda_k > \gamma$, we have

$$\langle J''(0, 0)(0, \phi_k), (0, \phi_k)\rangle = \int_\Omega (\Delta\phi_k \cdot \phi_k + \gamma\phi_k^2)dx = (-\lambda_k + \gamma)\int_\Omega \phi_k^2 dx < 0.$$

Since there are infinite many such pairs, it follows that the negative eigenspace of $J''(0, 0)$ is infinite-dimensional. This implies that critical point $(0, 0)$ of J has infinite Morse index. Finally, a similar argument as above holds true for $-J$. ∎

For a general dual functional $J \in C^1(H, \mathbb{R})$ with gradient $\nabla J = (\frac{\partial J}{\partial u}, \frac{\partial J}{\partial v})$, we define a solution manifold $\widetilde{\mathcal{M}}$ of J in H by

$$\widetilde{\mathcal{M}} = \left\{(u, v) \in H : \frac{\partial J}{\partial u} \perp u, \frac{\partial J}{\partial v} \perp v\right\}. \tag{5.14}$$

Clearly, $\widetilde{\mathcal{M}}$ contains all critical points of J in H.

Proposition V.4 *For J in (5.13), there holds*

$$J(u, v) = \int_\Omega \Big[(\frac{1}{2} - \frac{1}{p+1})|u|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v|^{q+1}\Big] dx \geq 0, \ \forall (u, v) \in \widetilde{\mathcal{M}}.$$

Consequently, $(0, 0) \in \widetilde{\mathcal{M}}$ is the least energy saddle point of J with $J(0, 0) = 0$.

Proof. For every point $(u, v) \in \widetilde{\mathcal{M}}$, Definition II.1 and equation (5.12) show that

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega (-\Delta u \cdot u) dx = \int_\Omega \left[\lambda u^2 - \delta uv + |u|^{p+1}\right] dx, \tag{5.15}$$

$$\int_\Omega |\nabla v|^2 dx = \int_\Omega (-\Delta v \cdot v) dx = \int_\Omega \left[\gamma v^2 + \delta uv - |v|^{q+1}\right] dx, \tag{5.16}$$

which together with (5.13) leads to

$$\begin{aligned} J(u, v) &= \frac{1}{2}\int_\Omega \Big[(-\Delta u \cdot u - (-\Delta v \cdot v)) - (\lambda u^2 - 2\delta uv - \gamma v^2)\Big] dx - \int_\Omega G(u, v) dx \\ &= \frac{1}{2}\int_\Omega \Big[(\lambda u^2 - \delta uv + |u|^{p+1}) - (\gamma v^2 + \delta uv - |v|^{q+1})\Big] dx \\ &\quad -\frac{1}{2}\int_\Omega (\lambda u^2 - 2\delta uv - \gamma v^2) dx - \int_\Omega G(u, v) dx \\ &= \frac{1}{2}\int_\Omega (|u|^{p+1} + |v|^{q+1}) dx - \int_\Omega (\frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1}) dx \\ &= \int_\Omega \Big[(\frac{1}{2} - \frac{1}{p+1})|u|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v|^{q+1}\Big] dx \geq 0 \text{ (since } p, q > 1). \quad \blacksquare \end{aligned}$$

Next, if denoting by σ_1 the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, then we have

Proposition V.5 *If $\gamma < \sigma_1$, then for any critical point $(\bar{u}, \bar{v}) \neq (0,0)$ of J in (5.13), there hold (i) $\bar{u} \neq 0, \bar{v} \neq 0$; (ii) $\int_\Omega \bar{u}\bar{v}dx > 0$.*

Proof. (i) is trivial. For (ii), since $(\bar{u}, \bar{v})$ is a critical point of J, then $(\bar{u}, \bar{v})$ is a weak solution of (5.12). Multiplying the second equation in (5.12) by $\bar{v}$ yields

$$\int_\Omega |\nabla \bar{v}|^2 dx = \langle -\Delta \bar{v}, \bar{v} \rangle = \delta \int_\Omega \bar{u}\bar{v}dx + \gamma \int_\Omega \bar{v}^2 dx - \int_\Omega |\bar{v}|^{q+1} dx,$$

or equivalently

$$\delta \int_\Omega \bar{u}\bar{v}dx = \int_\Omega |\nabla \bar{v}|^2 dx - \gamma \int_\Omega \bar{v}^2 dx + \int_\Omega |\bar{v}|^{q+1} dx$$

from which, together with (i), assertion (ii) follows via the Poincare inequality. ∎

b. Properties of the L-$\perp$ Selection

Still, we denote by σ_1 the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$. The lemma below confirms the existence and differentiability of an L-$\perp$ selection function p of J in (5.13).

Lemma V.1 *Assume $\gamma \leq \lambda < \sigma_1$. For every unit vector $(\bar{u}, \bar{v})$ with $\int_\Omega \bar{u}\bar{v}dx \neq 0$, there exists a differentiable local peak selection p of J in (5.13) w.r.t. $L = \{0\} \times \{0\}$ around $(\bar{u}, \bar{v})$ such that $p(\bar{u}, \bar{v}) = (\bar{t}\bar{u}, \bar{s}\bar{v})$ with $\frac{\bar{t}}{\bar{s}} \int_\Omega \bar{u}\bar{v}dx > 0$ for some $\bar{t}, \bar{s}$.*

Proof. In the following, for simplicity, we use $\int$ instead of $\int_\Omega$ to stand for the integral over Ω. By definition, an L-$\perp$ selection $p(\bar{u}, \bar{v}) = (t\bar{u}, s\bar{v})$ of J in (5.13) can be solved from the following equations

$$\frac{\partial J}{\partial t}(t\bar{u}, s\bar{v}) = t(\int [|\nabla \bar{u}|^2 - \lambda \bar{u}^2]dx) + \delta s \int \bar{u}\bar{v}dx - |t|^{p-1}t \int |\bar{u}|^{p+1}dx = 0 \quad (5.17)$$

$$\frac{\partial J}{\partial s}(t\bar{u}, s\bar{v}) = s(\int [\gamma \bar{v}^2 - |\nabla \bar{v}|^2]dx) + \delta t \int \bar{u}\bar{v}dx - |s|^{q-1}s \int |\bar{v}|^{q+1}dx = 0 \quad (5.18)$$

for a nontrivial solution (t, s) (i.e., $ts \neq 0$).

For notational convenience, denote $a_0 = \delta \int \bar{u}\bar{v}dx, a_1 = \int [|\nabla \bar{u}|^2 - \lambda \bar{u}^2]dx$, $a_2 = \int |\bar{u}|^{p+1}dx, b_1 = \int [|\nabla \bar{v}|^2 - \gamma \bar{v}^2]dx, b_2 = \int |\bar{v}|^{q+1}dx$. Clearly, by assumptions, $a_i > 0, b_i > 0, i = 1, 2$. Then, (5.18) gives

$$t = \frac{b_1 s + b_2 |s|^{q-1} s}{a_0}. \tag{5.19}$$

Since we seek nonzero solutions of (5.17) and (5.18), substituting (5.19) into (5.17) and dividing the resulting equation by s yields

$$(b_1 + b_2|s|^{q-1})\frac{a_1}{a_0} + a_0 - \left|\frac{b_1 s + b_2|s|^{q-1}s}{a_0}\right|^{p-1} \frac{(b_1 + b_2|s|^{q-1})}{a_0} a_2 = 0. \tag{5.20}$$

Define

$$\psi(s) = (b_1 + b_2|s|^{q-1})\frac{a_1}{a_0} + a_0 - \left|\frac{b_1 s + b_2|s|^{q-1}s}{a_0}\right|^{p-1} \frac{(b_1 + b_2|s|^{q-1})}{a_0} a_2, \forall s \in [0, \infty).$$

Clearly, ψ is a continuous function with $\psi(0) = \frac{b_1 a_1}{a_0} + a_0$ and $\psi(s) \approx -|s|^{pq-1} \frac{b_2^p a_2}{|a_0|^{p-1} a_0}$ (when s is sufficiently large). We then see that $\psi(0)\psi(\infty) < 0$ because a_1, a_2, b_1 are all positive. Thus, by the mean value theorem, there exists $\bar{s} > 0$ such that $\psi(\bar{s}) = 0$. Substituting $\bar{s}$ into (5.19) gives $\bar{t} = \frac{(b_1 + b_2|\bar{s}|^{q-1})\bar{s}}{a_0} \neq 0$ since $b_1 + b_2|\bar{s}|^{q-1} > 0$. Thus,

$$\frac{\bar{t}}{\bar{s}}\delta \int \bar{u}\bar{v}dx = \frac{\bar{t}}{\bar{s}} a_0 = b_1 + b_2|\bar{s}|^{q-1} > 0 \quad \text{or} \quad \frac{\bar{t}}{\bar{s}} \int \bar{u}\bar{v}dx > 0 \text{ since } \delta > 0. \tag{5.21}$$

Next, we show that $p(\bar{u}, \bar{v}) = (\bar{t}\bar{u}, \bar{s}\bar{v})$ is a local maximum of J in the subspace $\text{span}\{\bar{u}\} \times \text{span}\{\bar{v}\}$; i.e., we verify that the Hessian matrix

$$\begin{aligned} Q &= \left.\begin{pmatrix} \frac{\partial^2}{\partial t^2} J(t\bar{u}, s\bar{v}) & \frac{\partial^2}{\partial t \partial s} J(t\bar{u}, s\bar{v}) \\ \frac{\partial^2}{\partial s \partial t} J(t\bar{u}, s\bar{v}) & \frac{\partial^2}{\partial s^2} J(t\bar{u}, s\bar{v}) \end{pmatrix}\right|_{(t,s)=(\bar{t},\bar{s})} \\ &= \begin{pmatrix} a_1 - a_2 p|\bar{t}|^{p-1} & a_0 \\ a_0 & -b_1 - b_2 q|\bar{s}|^{q-1} \end{pmatrix} \end{aligned} \tag{5.22}$$

is negative definite. Since $(\bar{t},\bar{s})$ solves (5.17)-(5.18), we have

$$a_1 = -\frac{\bar{s}}{\bar{t}}a_0 + a_2|\bar{t}|^{p-1},\ b_1 = \frac{\bar{t}}{\bar{s}}a_0 - b_2|\bar{s}|^{q-1}. \tag{5.23}$$

Substituting (5.23) into (5.22) gives

$$Q = \begin{pmatrix} -\frac{\bar{s}}{\bar{t}}a_0 - a_2(p-1)|\bar{t}|^{p-1} & a_0 \\ a_0 & -\frac{\bar{t}}{\bar{s}}a_0 - b_2(q-1)|\bar{s}|^{q-1} \end{pmatrix}. \tag{5.24}$$

Since $a_2, b_2 > 0$, $p, q > 1$, (5.21) implies that the diagonal elements of Q are negative and the determinant $|Q| > a_2b_2(p-1)(q-1)|\bar{t}|^{p-1}|\bar{s}|^{q-1} > 0$, from which it follows that Q is negative definite. Consequently, $p(\bar{u},\bar{v}) = (\bar{t}\bar{u},\bar{s}\bar{v})$ is a local maximum of J in $\text{span}\{\bar{u}\}\times\text{span}\{\bar{v}\}$.

Finally, we show that such p can be extended locally as a differentiable local peak selection of J around $(\bar{u},\bar{v})$. Consider the following equations

$$\begin{cases} F_1(u,v,t,s) \equiv \frac{\partial J}{\partial t}(tu,sv) = 0 \\ F_2(u,v,t,s) \equiv \frac{\partial J}{\partial s}(tu,sv) = 0 \end{cases} \tag{5.25}$$

and define a matrix function

$$\mathcal{Q}(u,v,t,s) \equiv \frac{\partial(F_1,F_2)}{\partial(t,s)} = \begin{pmatrix} \frac{\partial^2}{\partial t^2}J(tu,sv) & \frac{\partial^2}{\partial t\partial s}J(tu,sv) \\ \frac{\partial^2}{\partial s\partial t}J(tu,sv) & \frac{\partial^2}{\partial s^2}J(tu,sv) \end{pmatrix}. \tag{5.26}$$

Obviously, $(\bar{u},\bar{v},\bar{t},\bar{s})$ solves (5.25) and $\mathcal{Q}\Big|_{(u,v,t,s)=(\bar{u},\bar{v},\bar{t},\bar{s})} = Q$.

Since $|Q| > 0$ (i.e., Q is invertible), by the implicit function theorem, there exists an open neighborhood $\mathcal{N}(\bar{u},\bar{v})$ of $(\bar{u},\bar{v})$ such that for every $(u,v)\in\mathcal{N}(\bar{u},\bar{v})\cap S_{L^\perp}$, (5.25) can be uniquely solved for $(t(u,v),s(u,v))$, where $t(u,v),s(u,v)$ are differentiable functions of (u,v) with $(t(\bar{u},\bar{v}),s(\bar{u},\bar{v})) = (\bar{t},\bar{s})$. Hence a differentiable local L-$\perp$ selection p with $p(\bar{u},\bar{v}) = (\bar{t}\bar{u},\bar{s}\bar{v})$ is well-defined in $\mathcal{N}(\bar{u},\bar{v})\cap S_{L^\perp}$. With $J\in C^2$, it follows that $\mathcal{Q}(t(u,v),s(u,v)) = \mathcal{Q}(u,v,t,s)$ is continuous in $\mathcal{N}(\bar{u},\bar{v})\cap S_{L^\perp}$. Since

Q is strictly negative definite and $\mathcal{Q}(t(\bar{u},\bar{v}), s(\bar{u},\bar{v})) = Q$, we can conclude that $\mathcal{Q}(t(u,v), s(u,v))$ is strictly negative definite, $\forall (u,v) \in \mathcal{N}(\bar{u},\bar{v}) \cap S_{L^\perp}$. Therefore, such p is also a local peak selection of J w.r.t. L. The lemma is thus proved. ∎

In this preceding lemma, we see that under some appropriate assumptions an L-$\perp$ selection p of J w.r.t. $L = \{0\} \times \{0\}$ exists at least locally. For a general $L = L_1 \times L_2 \subset H$, we define a solution manifold $\mathcal{M}$ by

$$\mathcal{M} = \Big\{ p(u,v) \neq (0,0) : (u,v) \in S_{L^\perp} \Big\}.$$

In particular, for $L = \{0\} \times \{0\}$, denote the solution manifold $\mathcal{M}_0$ by

$$\mathcal{M}_0 = \Big\{ p(u,v) \neq (0,0) : \|(u,v)\| = 1 \Big\}.$$

Clearly, $\mathcal{M} \subseteq \mathcal{M}_0 \subseteq \widetilde{\mathcal{M}}, \forall L \subset H$, where $\widetilde{\mathcal{M}}$ is defined in (5.14). Here, the point $(0,0)$ is excluded from the solution manifold $\mathcal{M}$ or $\mathcal{M}_0$ because it is usually a trivial solution. Next, we continue to explore some properties of J in (5.13).

Theorem V.2 *Assume $\lambda < \sigma_1, \gamma < \sigma_1$. Then there exists a constant $\alpha > 0$ such that*

$$dist(\mathcal{M}_0, (0,0)) \geq \alpha > 0. \tag{5.27}$$

Consequently, $(0,0) \notin \overline{\mathcal{M}}_0$.

Proof. We start the proof by defining

$$\mathcal{M}' = \Big\{ p(u,v) \equiv (tu, sv) : tu \neq 0, \|(u,v)\| = 1 \Big\}.$$

Clearly, $\mathcal{M}' \subseteq \mathcal{M}_0$. We will prove that $\mathcal{M}_0 = \mathcal{M}'$ by verifying that $t\bar{u} = 0$ implies $s\bar{v} = 0$ for every L-$\perp$ selection $p(\bar{u},\bar{v}) = (t\bar{u}, s\bar{v})$ of J w.r.t. $\{0\} \times \{0\}$. For each unit vector $(\bar{u},\bar{v}) \in H$, assume $p(\bar{u},\bar{v})$ is an L-$\perp$ selection of J w.r.t. $\{0\} \times \{0\}$. By (5.18),

$t\bar{u} = 0$ gives

$$s\Big(\int (|\nabla \bar{v}|^2 - \gamma \bar{v}^2)dx + |s|^{q-1}\int |\bar{v}|^{q+1}dx\Big) = 0 \tag{5.28}$$

from which, it follows that either $s = 0$ or $\int(|\nabla\bar{v}|^2 - \gamma\bar{v}^2)dx + |s|^{q-1}\int|\bar{v}|^{q+1}dx = 0$. By the Poincare inequality, $\gamma < \sigma_1$ implies $\int(|\nabla\bar{v}|^2 - \gamma\bar{v}^2)dx > 0$, $\forall \bar{v} \neq 0$. Hence, $\int(|\nabla\bar{v}|^2 - \gamma\bar{v}^2)dx + |s|^{q-1}\int|\bar{v}|^{q+1}dx = 0$ if and only if $\bar{v} = 0$. Hence, $t\bar{u} = 0$ implies $s\bar{v} = 0$. Therefore, $p(u,v) \in \mathcal{M}'$ for every $p(u,v) \in \mathcal{M}_0$, i.e., $\mathcal{M}_0 \subseteq \mathcal{M}'$. Thus, $\mathcal{M}_0 = \mathcal{M}'$.

Now, for each $p(\bar{u},\bar{v}) = (t\bar{u}, s\bar{v}) \in \mathcal{M}_0 = \mathcal{M}'$, (5.17) gives

$$|t|^{p-1}\int |\bar{u}|^{p+1}dx = \int [|\nabla\bar{u}|^2 - \lambda\bar{u}^2]dx + \frac{s}{t}\delta\int \bar{u}\bar{v}dx. \tag{5.29}$$

Note that $\frac{s}{t}\delta\int \bar{u}\bar{v}dx \geq 0$ (Here, $\bar{v}$ is allowed to be a zero function) due to (5.21) wherein $\bar{t}, \bar{s}$ are replaced by t, s, respectively. Hence,

$$\begin{aligned} c_0|t|^{p-1}\Big(\int |\nabla\bar{u}|^2dx\Big)^{\frac{p+1}{2}} \geq |t|^{p-1}\int|\bar{u}|^{p+1}dx \geq \int[|\nabla\bar{u}|^2 - \lambda\bar{u}^2]dx \\ \geq (1 - \frac{\lambda}{\sigma_1})\int|\nabla\bar{u}|^2dx \end{aligned} \tag{5.30}$$

or equivalently

$$|t|\Big(\int|\nabla\bar{u}|^2dx\Big)^{1/2} \geq \Big(\frac{1}{c_0}(1 - \frac{\lambda}{\sigma_1})\Big)^{\frac{1}{p-1}} > 0 \tag{5.31}$$

for some constant $c_0 > 0$ independent of $\bar{u}$ via the Poincare and Sobolev inequalities.

Setting $\alpha = (\frac{1}{c_0} - \frac{\lambda}{c_0\sigma_1})^{\frac{1}{p-1}}$ gives

$$dist(p(\bar{u},\bar{v}),(0,0)) \geq |t|\Big(\int|\nabla\bar{u}|^2dx\Big)^{1/2} \geq \alpha > 0,\ \forall p(\bar{u},\bar{v}) \in \mathcal{M}_0,$$

from which it follows that

$$dist(\mathcal{M}_0,(0,0)) \geq \alpha > 0. \quad \blacksquare$$

The preceding theorem indicates that the only trivial solution $(0,0)$ to (5.12) is always separated away from $\overline{\mathcal{M}}_0$, the closure of the solution manifold $\mathcal{M}_0$. In view of this, once a solution to (5.12) is located on $\mathcal{M}_0$, it will be nontrivial.

When a functional J is positive or semi-positive definite, then by our local min-orthogonal characterization on saddle points (i.e., Theorem II.2), we can look for local minima of J on the solution manifold $\mathcal{M}$ (via the local min-orthogonal method developed in Chapter II) instead of seeking its finite Morse index saddle points in the entire space H. However, when the functional J is strongly indefinite, such local minima of J on $\mathcal{M}$ may not exist and hence the local min-orthogonal method will fail to find its saddle points of infinite Morse index. The next theorem confirms this observation.

Theorem V.3 *Let $L = L_1 \times L_2 \subset H$ with $\dim(L) < \infty$ and p be a differentiable L-$\perp$ selection of J in (5.13) w.r.t. L at $\bar{w} = (\bar{u}, \bar{v}) \in S_{L^\perp}$ such that $p_i(\bar{w}) \notin L_i$, $i = 1, 2$, where $p(\bar{w}) = (p_1(\bar{w}), p_2(\bar{w}))$. If, in addition, $||\nabla J(p(\bar{w}))|| = 0$, then $\bar{w}$ is a saddle point of $J(p(\cdot))$ on $S_{L^\perp}$. Consequently, $p(\bar{w})$ is a saddle point of J on $\mathcal{M}$.*

Proof. Clearly, $\bar{w}$ is a critical point of $J(p(\cdot))$ on $S_{L^\perp}$. So we only need to show that $\bar{w}$ is a saddle point of $J(p(\cdot))$ on $S_{L^\perp}$. Suppose by contradiction $\bar{w}$ is a minimum of $J(p(\cdot))$ on $S_{L^\perp}$. Then, Theorem IV.4 or Lemma IV.5 shows that $MI(p(\bar{w})) \leq \dim(L) + 2 < \infty$. This obviously contradicts Prop. V.3. Finally, $\bar{w}$ is a saddle point of $J(p(\cdot))$ on $S_{L^\perp}$ implies that $p(\bar{w})$ is a saddle point of $J(\cdot)$ on $\mathcal{M}$ since p is differentiable at $\bar{w}$. ∎

Theorem V.3 partially answers the question that saddle points of infinite Morse index usually cannot be approximated by the local min-orthogonal method developed in Chapter II since the support L has to be finite-dimensional in numerical implementation. This in turn motivates us to develop the computational theory and methods for finding critical points of strongly indefinite functionals. Finally, it is easy to see that the conclusions in Theorem V.3 also hold true for more general strongly indefinite functionals.

c. Numerical Results

To find numerical solutions to (5.12), we choose $p = q = 3$, $\lambda = \gamma = -0.5, \delta = 5$ and use two different domains: a square $\Omega_1 = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ and a disk $\Omega_2 = \{x \in \mathbb{R}^2 : |x| < 1.4\}$. For each $(u, v) \in H$, we compute the corresponding gradient $d \equiv (d_1, d_2) = \nabla J(u, v)$ of J in (5.13) as follows.

Since for every $\phi = (\phi_1, \phi_2) \in H$ we have

$$\begin{aligned}
\langle d, \phi\rangle_H &= \int_\Omega (\nabla d_1 \nabla\phi_1 + \nabla d_2 \nabla\phi_2)\, dx \\
&= -\int_\Omega (\Delta d_1 \phi_1 + \Delta d_2 \phi_2)\, dx \\
&\equiv \tfrac{d}{dt}|_{t=0} J((u, v) + t\phi) \\
&= \int_\Omega [(-\Delta u - \lambda u + \delta v - |u|^{p-1}u)\phi_1 + (\Delta v + \delta u + \gamma v - |v|^{q-1}v)\phi_2]\, dx,
\end{aligned} \tag{5.32}$$

it then follows that d solves the following linear elliptic system

$$\begin{cases}
\Delta d_1 = \Delta u + \lambda u - \delta v + |u|^{p-1}u & x \in \Omega, \\
\Delta d_2 = -(\Delta v + \delta u + \gamma v - |v|^{q-1}v) & x \in \Omega, \\
d_1 = d_2 = 0 & x \in \partial\Omega.
\end{cases} \tag{5.33}$$

Therefore, using the subroutine ASSEMPDE in MATLAB or other finite-element linear solvers, we can find the gradient d.

In our experiments, 32768 (resp. 18432) triangles were used for the square domain Ω_1 (resp. the disk domain Ω_2). In both cases, the tolerance ϵ is 8×10^{-5}. Figs. 10-13 (resp. Figs. 14-16) display both the profiles (left) and contour plots (right) of the first few solutions to system (5.12) on Ω_1 (resp. Ω_2). Fig. 17 (resp. Fig. 18) shows the convergence of the energy gap $|J(p(w^{k,2})) - J(p(w^{k,1}))|$ (top), the gradient norm $\|d^k\|$ (top) and the energy $J(p(w^k))$ (bottom) on computing the positive solution, see also Fig. 10 (resp. Fig. 14), to system (5.12) on Ω_1 (resp. Ω_2). Here, k is the iteration number. The initial direction used is $u_0 = v_0 = (1 - x_1^2)(1 - x_2^2)$ (resp. $u_0 = v_0 = (1.4^2 - x_1^2 - x_2^2)$) with $x = (x_1, x_2) \in \Omega_1$ (resp. $x = (x_1, x_2) \in \Omega_2$). From Figs. 17 and 18, one sees for both cases that $|J(p(w^{k,2})) - J(p(w^{k,1}))| = O(\|d^k\|^2)$ at the final iteration. This agrees with the gradient estimate established in Corollary III.1.

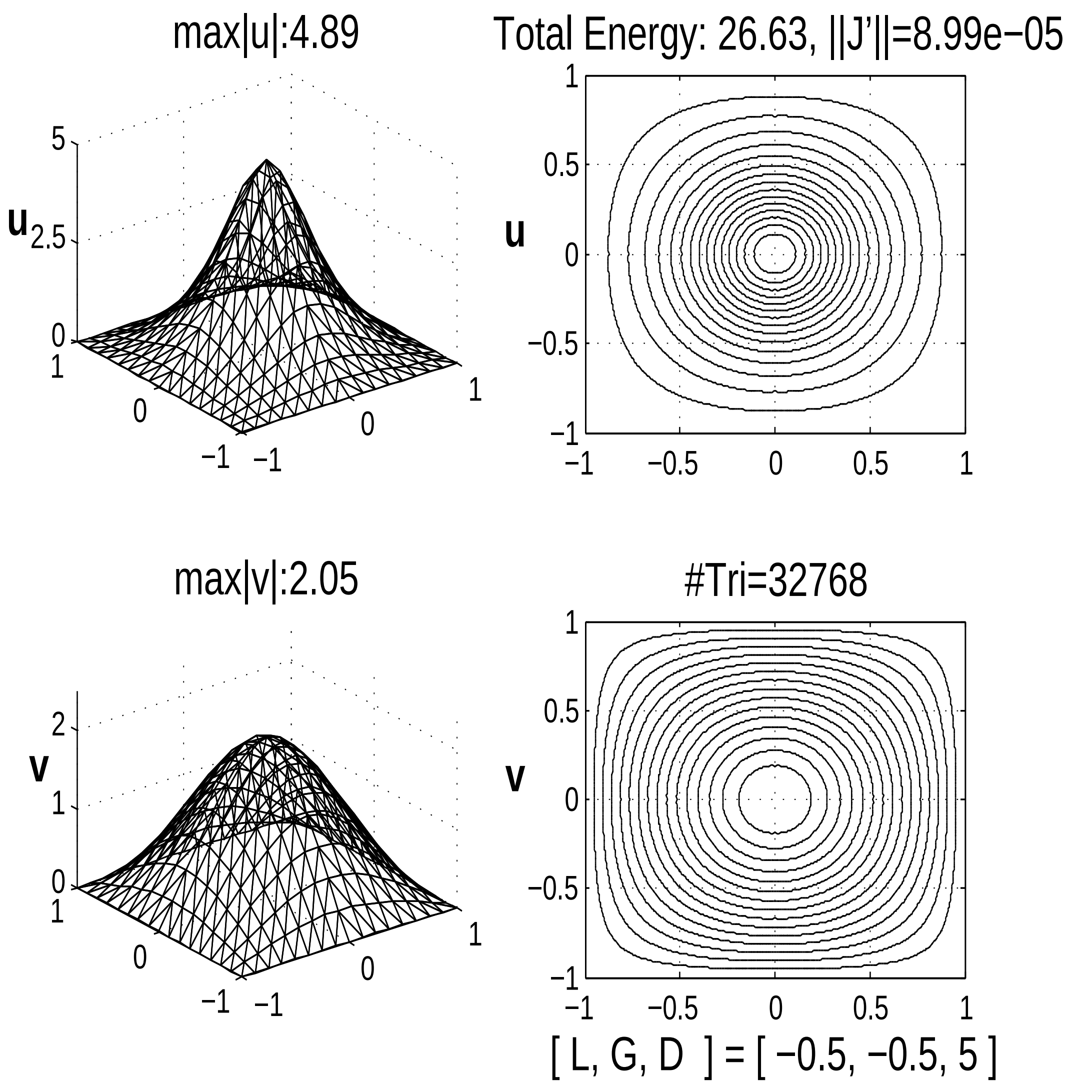

Fig. 10. A positive solution to (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$: profiles (left), contours (right).

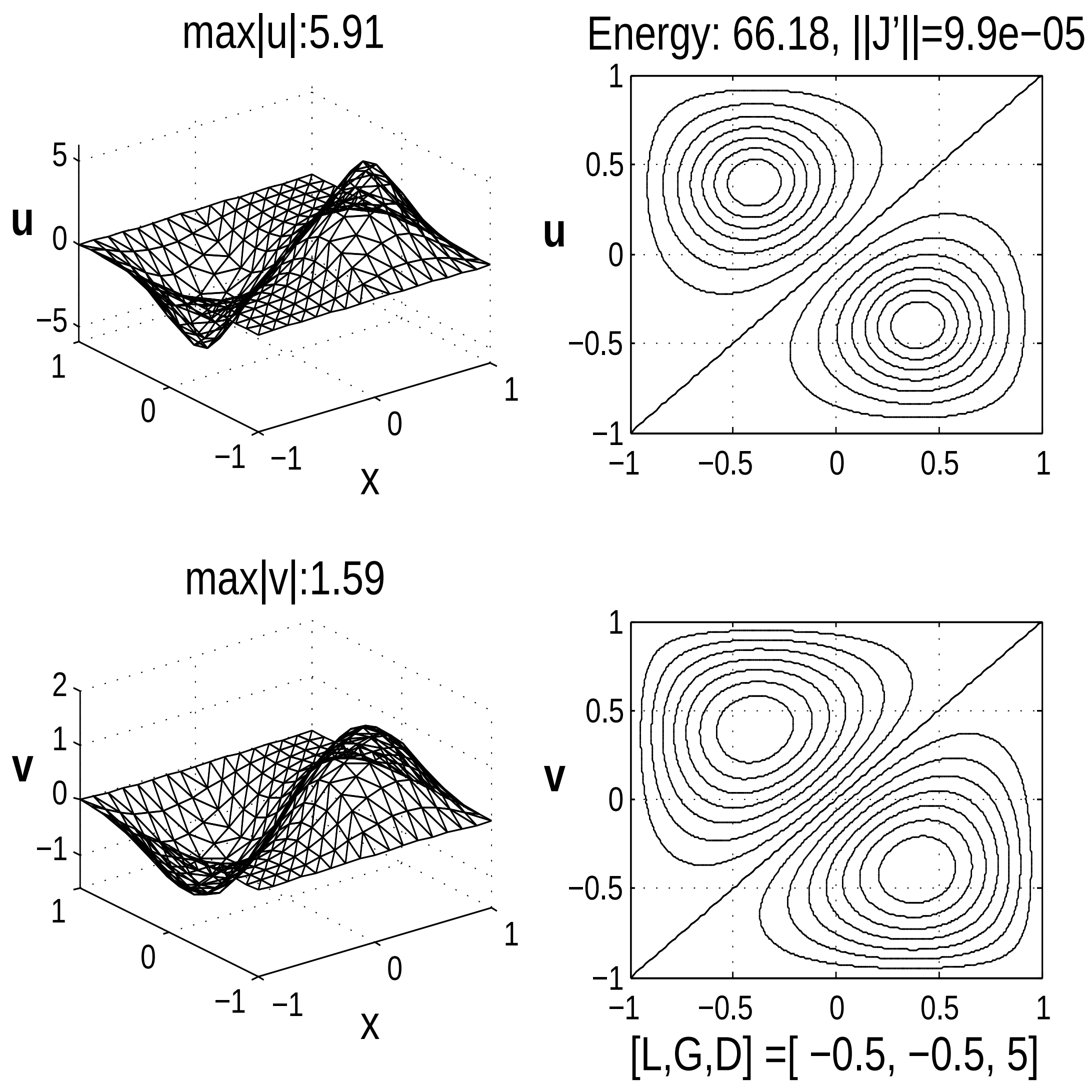

Fig. 11. A sign-changing solution to (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$: profiles (left), contours (right).

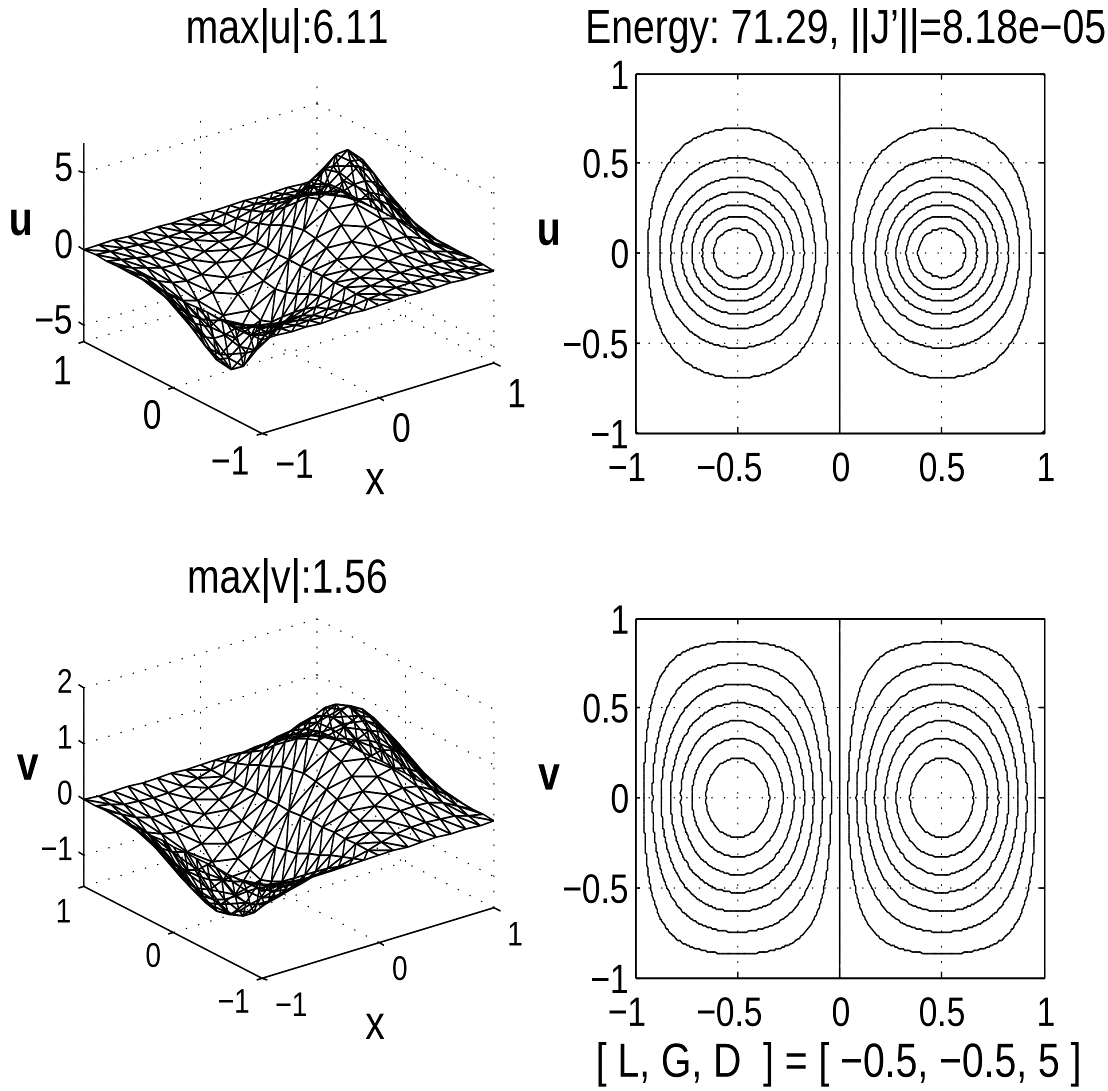

Fig. 12. A sign-changing 2-peak solution to (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$: profiles (left), contours (right).

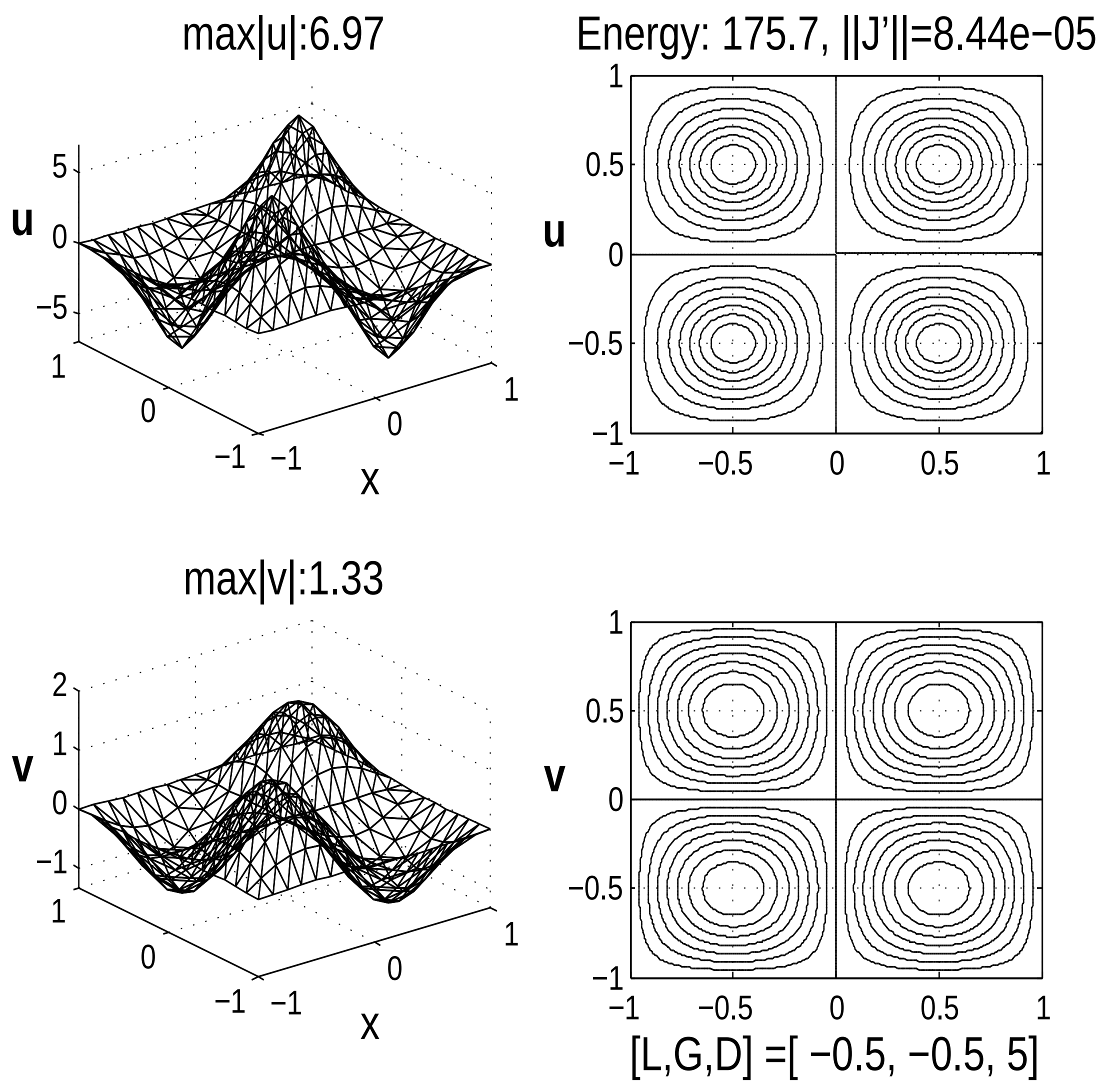

Fig. 13. A sign-changing 4-peak solution to (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$: profiles (left), contours (right).

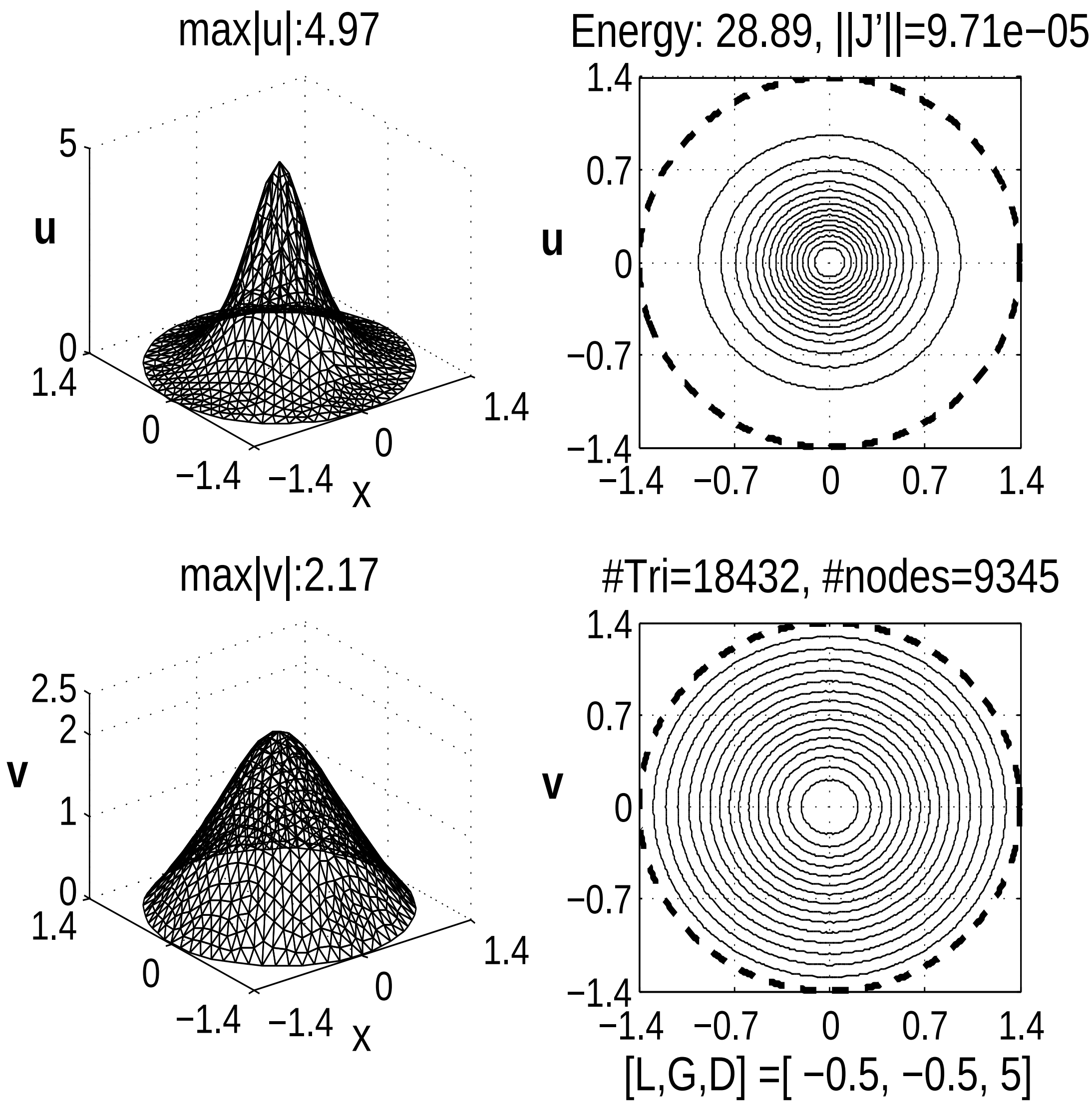

Fig. 14. A radial positive solution to (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$, $\Omega = \{x \in \mathbb{R}^2 : |x| < 1.4\}$: profiles (left), contours (right). The dashed circle indicates the domain boundary.

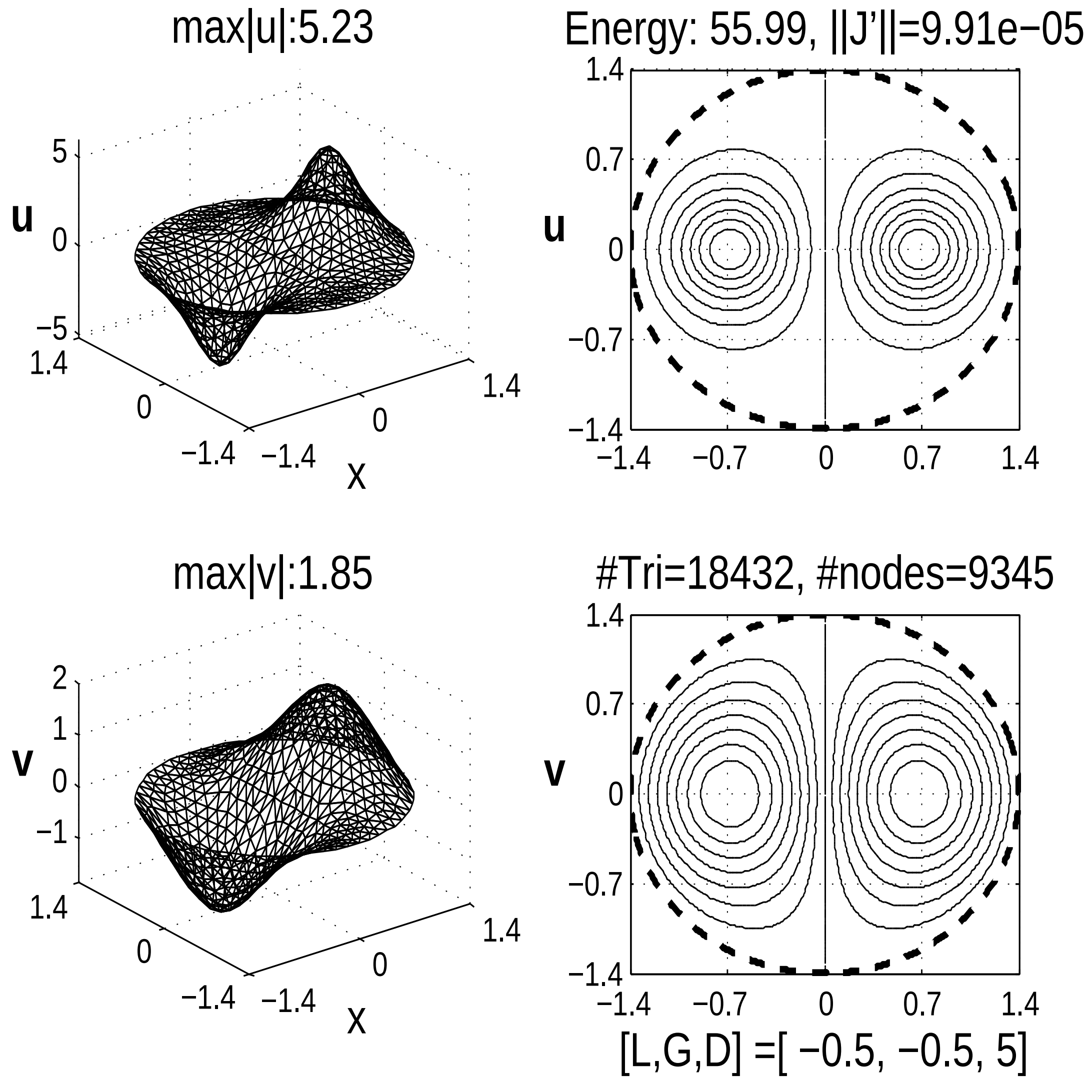

Fig. 15. A sign-changing 2-peak solution to (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$, $\Omega = \{x \in \mathbb{R}^2 : |x| < 1.4\}$: profiles (left), contours (right). The dashed circle indicates the domain boundary.

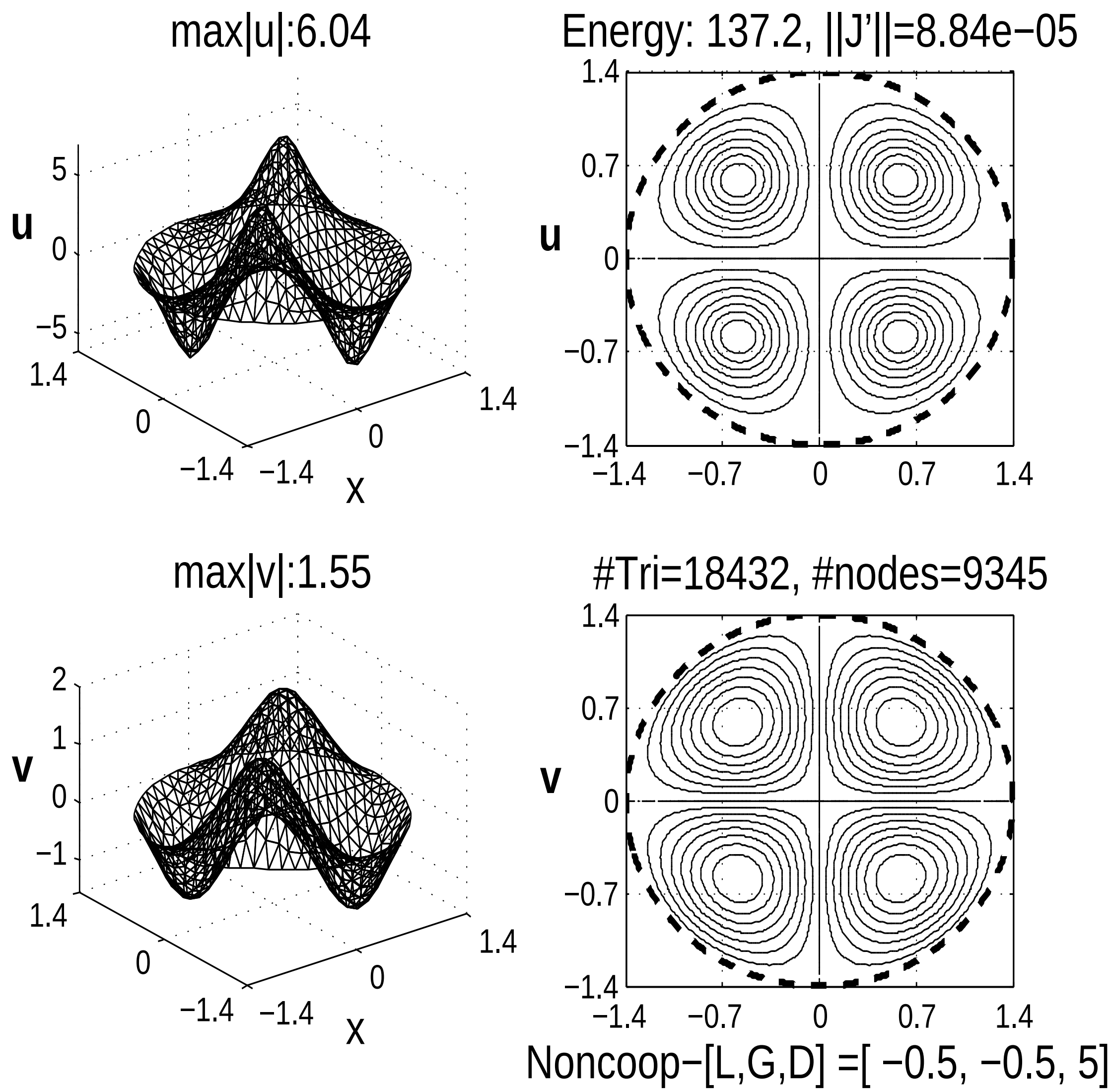

Fig. 16. A sign-changing 4-peak solution to (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$, $\Omega = \{x \in \mathbb{R}^2 : |x| < 1.4\}$: profiles (left), contours (right). The dashed circle indicates the domain boundary.

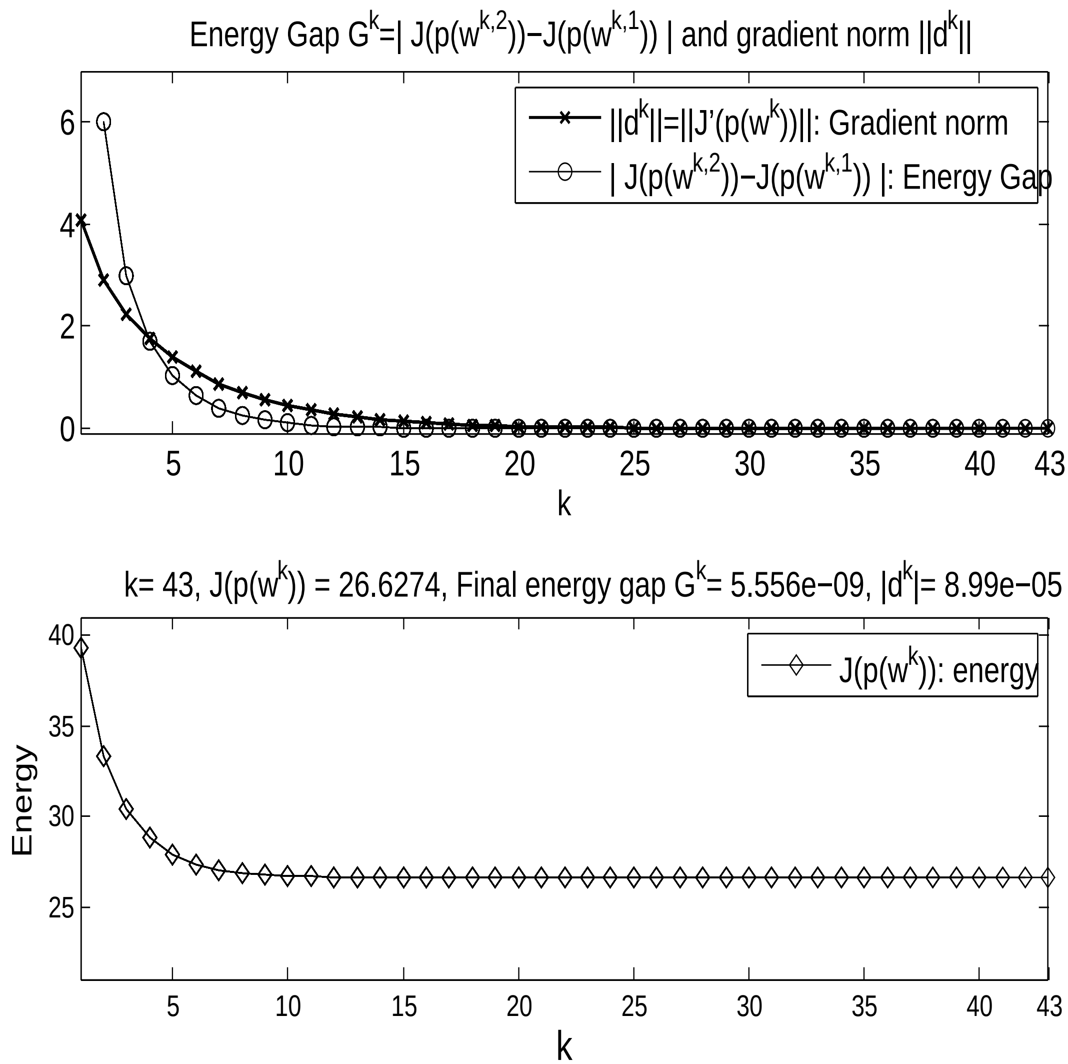

Fig. 17. Convergence test on the positive solution (depicted in Fig. 10) to system (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5, \Omega = (-1, 1)^2$: convergence of the energy gap $|J(p(w^{k,2})) - J(p(w^{k,1}))|$ (top), the gradient $\|d^k\|$ (top), and the energy $J(p(w^k))$ (bottom).

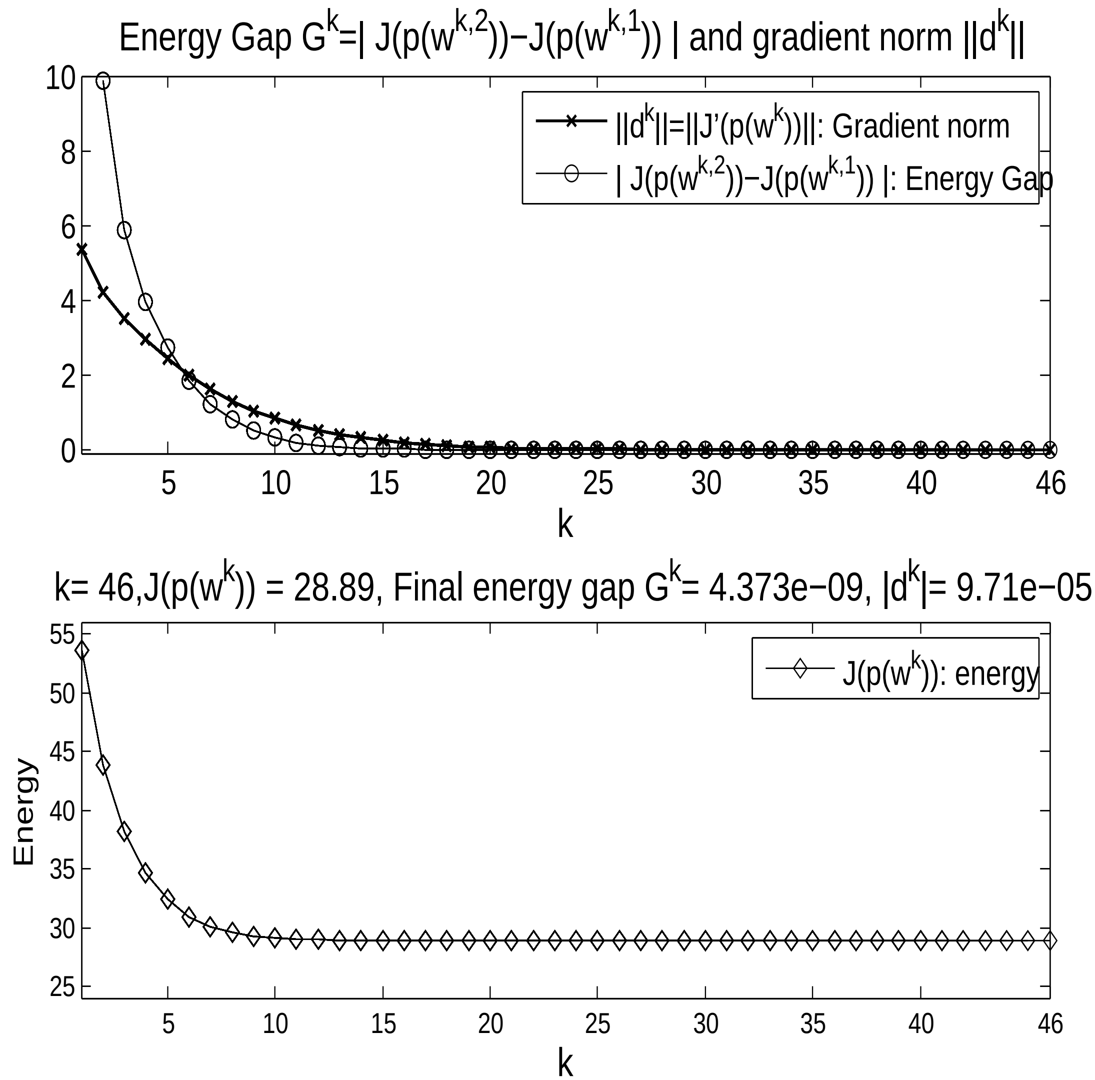

Fig. 18. Convergence test on the positive solution (depicted in Fig. 14) to system (5.12) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5, \Omega = \{x \in \mathbb{R}^2 : |x| < 1.4\}$: convergence of the energy gap $|J(p(w^{k,2})) - J(p(w^{k,1}))|$ (top), the gradient $\|d^k\|$ (top), and the energy $J(p(w^k))$ (bottom).

2. Indefinite Type

In this section we consider a noncooperative system for which the nonlinear term $G(x;u,v)$ is indefinite (i.e., neither bounded from below nor from above). Due to this indefinite nature, none of the existence results in [7,20,21,30,68] is applicable. Meanwhile, we have numerically found several solutions to a noncooperative system in which the nonlinear term $G(x;u,v)$ is indefinite.

Example V.3 *Choose $N=2$ (i.e., $\Omega \subset \mathbb{R}^2$) and $G(x;u,v) \equiv G(u,v) = \frac{1}{p+1}|u|^{p+1} - \frac{1}{q+1}|v|^{q+1}$ with $p,q>1$. System (5.11) becomes*

$$\begin{cases} -\Delta u = \lambda u - \delta v + |u|^{p-1}u & x \in \Omega, \\ -\Delta v = \delta u + \gamma v + |v|^{q-1}v & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \tag{5.34}$$

to which the associated energy functional

$$J(u,v) = \frac{1}{2}\int_\Omega \Big[(|\nabla u|^2 - |\nabla v|^2) - (\lambda u^2 - 2\delta uv - \gamma v^2)\Big]dx - \int_\Omega G(u,v)dx \tag{5.35}$$

is well-defined in $H = H_0^1(\Omega) \times H_0^1(\Omega)$.

One sees that for this particular example, both the asymptotic noncrossing $(F_4^{\pm})$ and crossing conditions (F_5)-(F_6) stated in the previous section fail due to

$$\liminf_{|(u,v)|\to\infty} \frac{2G(x;u,v)}{|(u,v)|^2} = -\infty, \ \limsup_{|(u,v)|\to\infty} \frac{2G(x;u,v)}{|(u,v)|^2} = \infty \text{ and } \limsup_{|(u,v)|\to 0} \frac{2G(x;u,v)}{|(u,v)|^2} = 0.$$

Proposition V.6 *Every critical point of J in (5.35) has infinite Morse index.*

Proposition V.7 *For every critical point (u,v) of J in (5.35), there holds*

$$J(u,v) = \int_\Omega \Big[(\frac{1}{2} - \frac{1}{p+1})|u|^{p+1} + (\frac{1}{q+1} - \frac{1}{2})|v|^{q+1}\Big]dx.$$

Proof. Refer to the proof of Prop. V.4. ▮

For this type of noncooperative system, however, we cannot give a general result on the existence of a local L-$\perp$ selection $\bar{p}$ (which eventually boils down to the existence of nontrivial solutions to a system of nonlinear algebraic equations and hence is very difficult to solve); instead, similar to Theorem V.2, we establish a separation result for the case $L = \{0\} \times \{0\}$. As before, let σ_1 be the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and define the associated solution manifold $\mathcal{M}_0$ by

$$\mathcal{M}_0 = \Big\{\bar{p}(u,v) \neq (0,0) : \|(u,v)\| = 1\Big\}.$$

Theorem V.4 *Assume $\delta > 0, \lambda, \gamma < \sigma_1$. Then there exists some $\alpha > 0$ such that*

$$\mathit{dist}(\mathcal{M}_0, (0,0)) \geq \alpha > 0. \tag{5.36}$$

Consequently, $(0,0) \notin \overline{\mathcal{M}_0}$.

Proof. For convenience, we still use the symbol $\int$ without the subscript Ω to stand for the integral over Ω and borrow some notations from Lemma V.1, namely,

$$a_0 = \delta \int \bar{u}\bar{v}dx,\ a_1 = \int [|\nabla \bar{u}|^2 - \lambda \bar{u}^2]dx,\ a_2 = \int |\bar{u}|^{p+1}dx,$$

$$b_1 = \int [|\nabla \bar{v}|^2 - \gamma \bar{v}^2]dx,\ b_2 = \int |\bar{v}|^{q+1}dx$$

for every unit vector $(\bar{u}, \bar{v}) \in H$. By definition, if $\bar{p}$ is an L-$\perp$ selection of J in (5.35) with respect to $L = \{0\} \times \{0\}$, then $\bar{p}(\bar{u}, \bar{v}) = (t\bar{u}, s\bar{v})$ with $(0,0) \neq (t,s) \in \mathbb{R}^2$ satisfying the following equations

$$\frac{\partial J}{\partial t}(t\bar{u}, s\bar{v}) = ta_1 + sa_0 - |t|^{p-1}ta_2 = 0, \tag{5.37}$$

$$\frac{\partial J}{\partial s}(t\bar{u}, s\bar{v}) = ta_0 - sb_1 + |s|^{q-1}sb_2 = 0. \tag{5.38}$$

Thus, it suffices to prove that $\exists\,\alpha > 0$ s.t. $\|\bar{p}(\bar{u},\bar{v})\| = \|(t\bar{u}, s\bar{v})\| \geq \alpha$, for any $\bar{p}(\bar{u},\bar{v}) \in \mathcal{M}_0$.

We have two cases: (i) $a_0 = 0$ (i.e., $\int \bar{u}\bar{v}dx = 0$), (ii) $a_0 \neq 0$ (i.e., $\int \bar{u}\bar{v}dx \neq 0$).

Case (i): $\int \bar{u}\bar{v}dx = 0$. For this case, one can see that equations (5.37) and (5.38) are actually decoupled. The fact $\|(\bar{u},\bar{v})\| = 1$ implies that at least one component of the vector $(\bar{u},\bar{v})$ must be nonzero. Then, using the Poincare and Sobolev inequalities (refer also to the lines of the proof to Lemma V.1), one can prove (5.36).

Case (ii): $\int \bar{u}\bar{v}dx \neq 0$. Clearly, $\bar{u} \neq 0, \bar{v} \neq 0$. Thus, $a_2, b_2 > 0$. By the Poincare inequality, $a_1 > 0, b_1 > 0$. Multiplying (5.37) by t and (5.38) by s and then subtracting one equation from another yields

$$t^2 a_1 - |t|^{p+1} a_2 + s^2 b_1 - |s|^{q+1} b_2 = 0 \tag{5.39}$$

or

$$t^2 a_1 + s^2 b_1 = |t|^{p+1} a_2 + |s|^{q+1} b_2. \tag{5.40}$$

Denote $\|\bar{u}\|_2 = (\int |\nabla \bar{u}|^2 dx)^{\frac{1}{2}}, \|\bar{v}\|_2 = (\int |\nabla \bar{v}|^2 dx)^{\frac{1}{2}}$. Then, applying the Poincare and Sobolev inequalities, we obtain

$$c_p |t|^{p+1} \|\bar{u}\|_2^{p+1} \geq |t|^{p+1} \int |\bar{u}|^{p+1} dx = |t|^{p+1} a_2, \tag{5.41}$$

$$c_q |s|^{q+1} \|\bar{v}\|_2^{q+1} \geq |s|^{q+1} \int |\bar{v}|^{q+1} dx = |s|^{q+1} b_2, \tag{5.42}$$

and

$$t^2 a_1 = t^2 \int [|\nabla \bar{u}|^2 - \lambda \bar{u}^2] dx \geq t^2 (1 - \frac{\lambda}{\sigma_1}) \|\bar{u}\|_2^2, \tag{5.43}$$

$$s^2 b_1 = t^2 \int [|\nabla \bar{v}|^2 - \gamma \bar{v}^2] dx \geq s^2 (1 - \frac{\gamma}{\sigma_1}) \|\bar{v}\|_2^2, \tag{5.44}$$

for some constants (independent of $\bar{u}, \bar{v}$) $c_p, c_q > 0$ which, with (5.40), leads to

$$\begin{aligned} c_p|t|^{p+1}\|\bar{u}\|_2^{p+1} + c_q|s|^{q+1}\|\bar{v}\|_2^{q+1} &\geq |t|^{p+1}a_2 + |s|^{q+1}b_2 \\ &\geq t^2(1 - \frac{\lambda}{\sigma_1})\|\bar{u}\|_2^2 + s^2(1 - \frac{\gamma}{\sigma_1})\|\bar{v}\|_2^2 \\ &\geq \min\left\{1 - \frac{\lambda}{\sigma_1}, 1 - \frac{\gamma}{\sigma_1}\right\} \|(t\bar{u}, s\bar{v})\|^2. \end{aligned} \tag{5.45}$$

With the fact $\|t\bar{u}\|_2^{p+1} \leq \|(t\bar{u}, s\bar{v})\|^{p+1}$ and $\|s\bar{v}\|_2^{q+1} \leq \|(t\bar{u}, s\bar{v})\|^{q+1}$, (5.45) shows

$$\begin{aligned} (c_p + c_q)(\|(t\bar{u}, s\bar{v})\|^{p+1} + \|(t\bar{u}, s\bar{v})\|^{q+1}) &\geq c_p\|t\bar{u}\|_2^{p+1} + c_q\|s\bar{v}\|_2^{q+1} \\ &\geq \min\left\{1 - \frac{\lambda}{\sigma_1}, 1 - \frac{\gamma}{\sigma_1}\right\} \|(t\bar{u}, s\bar{v})\|^2. \end{aligned} \tag{5.46}$$

Dividing both sides of the above inequality by $(c_p + c_q)\|(t\bar{u}, s\bar{v})\|^2$ gives

$$(\|(t\bar{u}, s\bar{v})\|^{p-1} + \|(t\bar{u}, s\bar{v})\|^{q-1}) \geq \frac{1}{c_p + c_q} \min\left\{1 - \frac{\lambda}{\sigma_1}, 1 - \frac{\gamma}{\sigma_1}\right\}.$$

Since $p, q > 1$ and $c_p, c_q, \sigma_1, \gamma, \lambda$ are independent of $\bar{u}, \bar{v}$, we conclude that $\exists \alpha > 0$ s.t.

$$\|\bar{p}(\bar{u}, \bar{v})\| = \|(t\bar{u}, s\bar{v})\| \geq \alpha, \forall \bar{p}(\bar{u}, \bar{v}) \in \mathcal{M}_0. \quad \blacksquare \tag{5.47}$$

Note that $(0, 0)$ is the only trivial solution to system (5.34). Therefore, as a consequence of the preceding theorem, once a solution is found on the solution manifold $\mathcal{M}_0$, it will be nontrivial. Finally, we end this section by presenting three solutions to system (5.34) with $\lambda = -0.5, \gamma = -1, \delta = 0.5, \Omega = (-2, 2) \times (-2, 2)$. Figs. 19-21 display the profiles (left) and contour plots (right) of the first three solutions to (5.34).

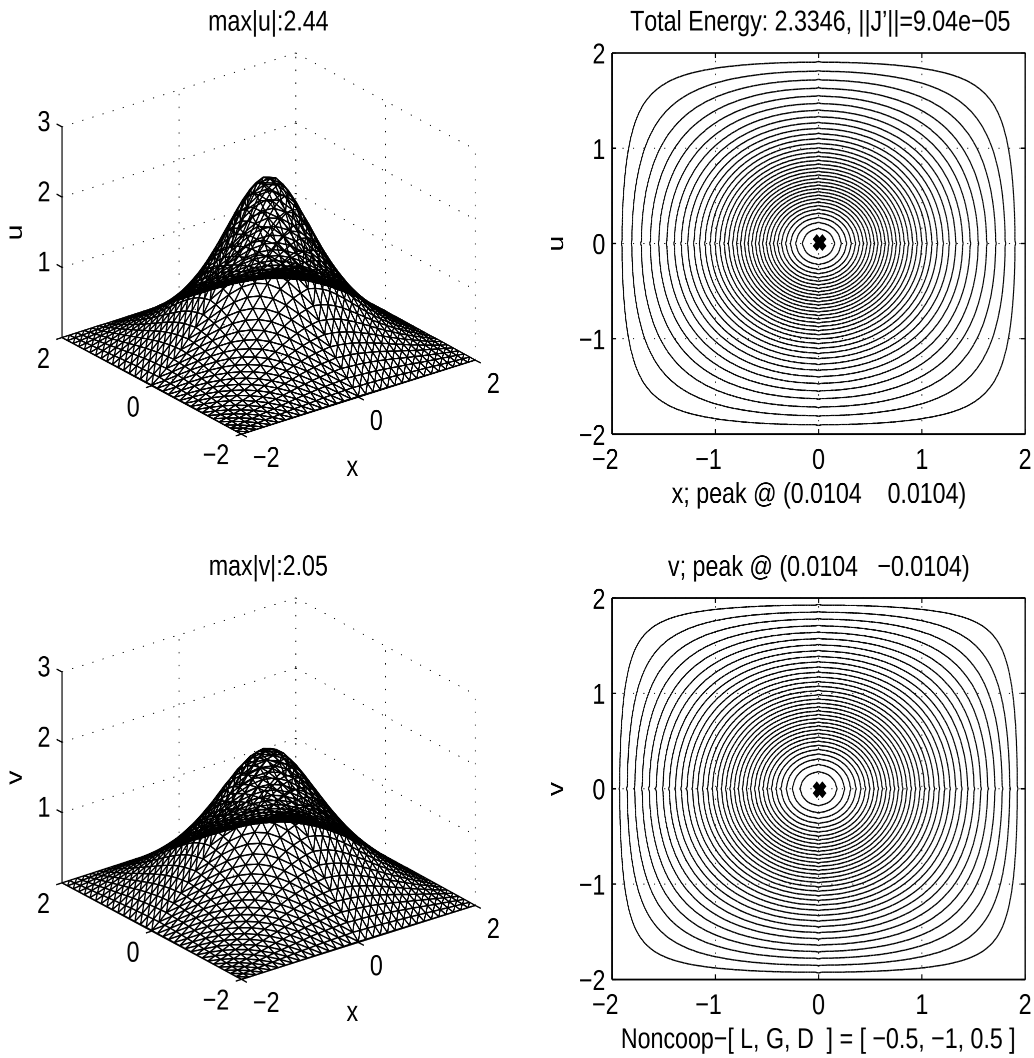

Fig. 19. A positive solution to (5.34) with $\Omega = (-2, 2) \times (-2, 2)$, $\lambda = -0.5$, $\gamma = -1$, $\delta = 0.5$: profiles (left), contours (right). The location of the positive peak is indicated by the symbol '$\times$'.

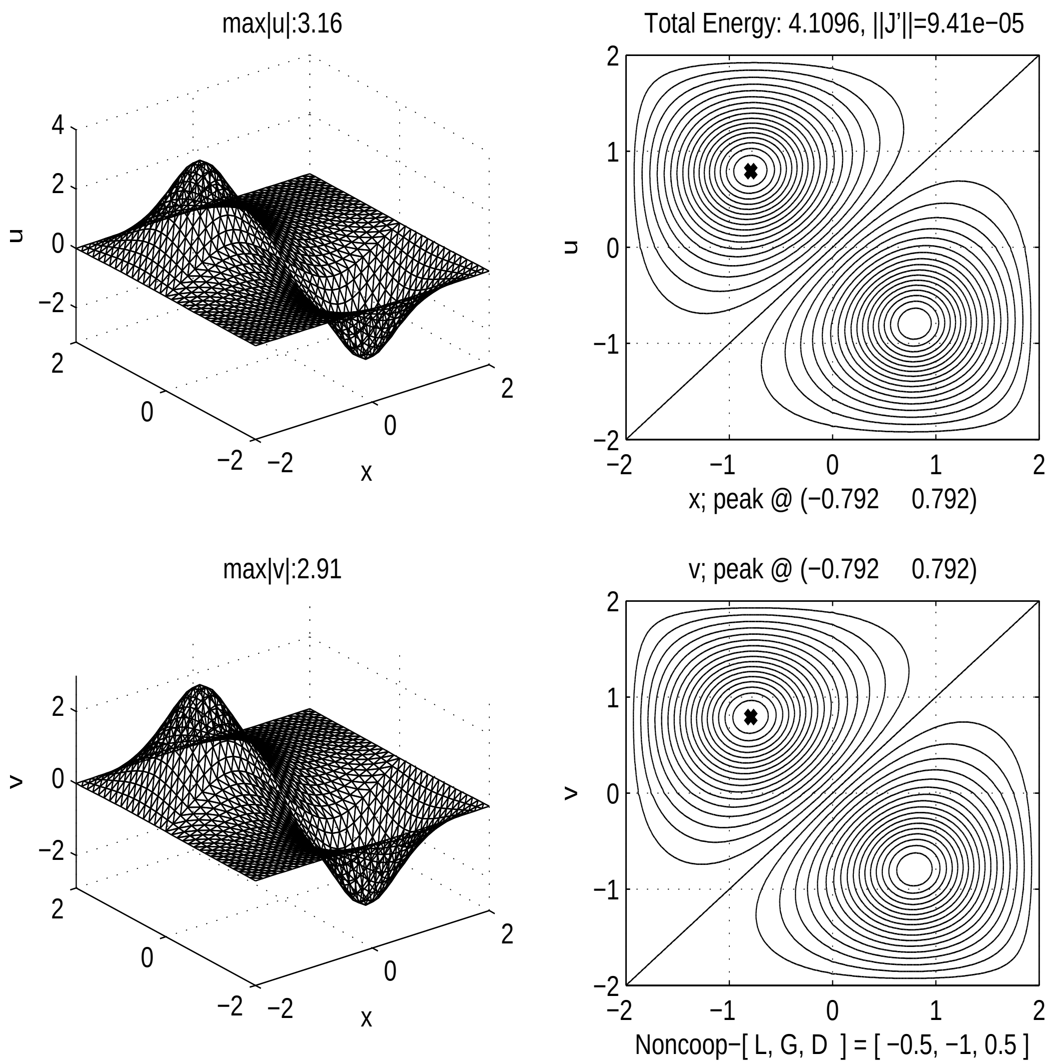

Fig. 20. A sign-changing solution to (5.34) with $\lambda = -0.5$, $\gamma = -1$, $\delta = 0.5$, $\Omega = (-2,2)\times(-2,2)$: profiles (left), contours (right). The location of its positive peak is indicated by the symbol '$\times$'.

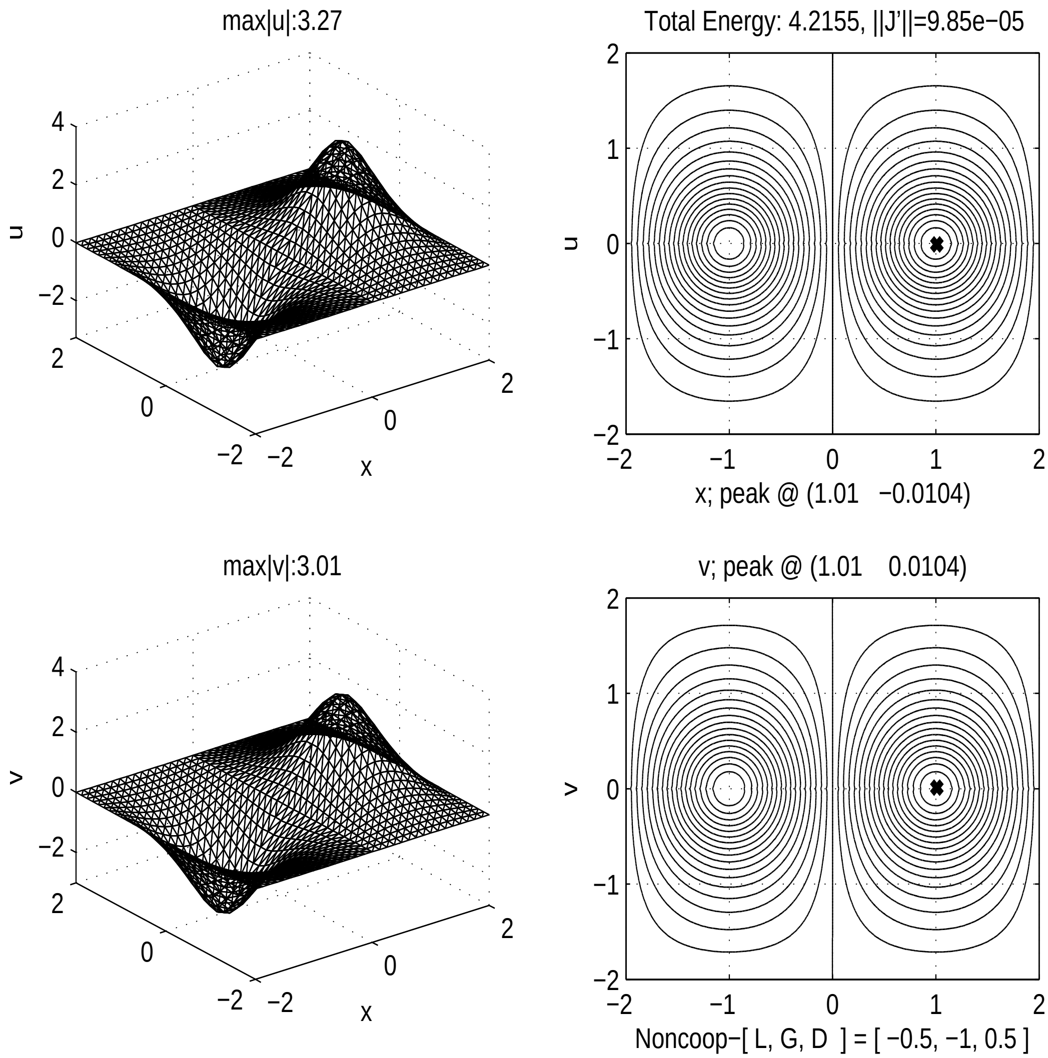

Fig. 21. A sign-changing solution to (5.34) with $\lambda = -0.5$, $\gamma = -1$, $\delta = 0.5$, $\Omega = (-2, 2) \times (-2, 2)$: profiles (left), contours (right). The location of its positive peak is marked by '$\times$'.

C. Hamiltonian Elliptic Systems

Consider Hamiltonian elliptic systems of the form

$$\begin{cases} -\Delta u = G_v(x;u,v) & x \in \Omega, \\ -\Delta v = G_u(x;u,v) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \tag{5.48}$$

where $\nabla G = (G_u, G_v)$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain. Systems of this type allow for a variational formulation [29,31,33], i.e., with some appropriate assumptions on G and a suitably chosen functional space H, solutions to (5.48) arise as critical points of the Lagrangian functional $J \in C^1(H, \mathbb{R})$ given by

$$J(u,v) = \int_\Omega \nabla u \nabla v dx - \int_\Omega G(x;u,v)dx. \tag{5.49}$$

For instance, one may assume the following hypotheses [29,31]

(F1) $G : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 in (u,v);

(F2) $\exists R > 0$ s.t. $\dfrac{1}{p+1}\dfrac{\partial G}{\partial u}(x;u,v)\cdot u + \dfrac{1}{q+1}\dfrac{\partial G}{\partial v}(x;u,v)\cdot v \geq G(x;u,v) > 0$ for all $(u,v) \in \mathbb{R}^2, |(u,v)| \geq R$ and $x \in \overline{\Omega}$ with $p,q > 0, 1/(p+1) + 1/(q+1) < 1$;

(F3) $\exists r > 0, a_1 > 0$ s.t. $G(x;u,v) \leq a_1(|u|^{p+1} + |v|^{q+1})$ if $|(u,v)| < r$;

(F4) $\exists a_2 > 0$ s.t. $|G_u(x;u,v)| \leq a_2(|u|^p + |v|^{p(q+1)/(p+1)} + 1)$,

$$|G_v(x;u,v)| \leq a_2(|v|^q + |u|^{q(p+1)/(q+1)} + 1).$$

As noted in [29,31], less restrictive assumptions on G can be made if $N = 1,2$.

It is easy to see that the quadratic part (i.e., the term $\int_\Omega \nabla u \nabla v dx$) in the functional J above is also strongly indefinite [31,33], i.e., there exists an orthogonal splitting $H = H^+ \oplus H^-$ with $\dim(H^\pm) = \infty$ such that it is positive definite in H^+ and negative definite in H^-.

System (5.48) admits a much richer structure and a rather different characterization due to the coupling of u, v in the term $\int_\Omega \nabla u \nabla v dx$. More precisely, there is no longer unique choice of function spaces for problem (5.48) because of a trade-off between u and v; for instance, if we demand more regularity on u, then we only need less regularity on v (so that $\int_\Omega \nabla u \nabla v dx$ is well-defined), and vice versa.

Next, we introduce some fractional Sobolev spaces so that the quadratic part $\int_\Omega \nabla u \nabla v dx$ of J in (5.49) can be well-defined. Let $\phi_n\,(n = 1, 2, ...)$ be normalized eigenfunctions of $-\Delta$ on $H_0^1(\Omega)$ with associated eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \uparrow \infty$. Assume $0 < s < 2$, then the operators $(-\Delta)^{s/2}$ can be defined by

$$(-\Delta)^{s/2} u = \sum_{k=1}^{\infty} \lambda_k^{s/2} \xi_k \phi_k, \ \forall u = \sum_{k=1}^{\infty} \xi_k \phi_k \in L^2(\Omega),$$

with domain

$$D((-\Delta)^{s/2}) = \left\{ \sum_{k=1}^{\infty} \xi_k \phi_k \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^s \xi_k^2 < \infty \right\}.$$

For any $t, s > 0$ with $s + t = 2$, let $E = E^s \times E^t$, where $E^s = D((-\Delta)^{s/2})$, $E^t = D((-\Delta)^{t/2})$. It is easy to verify that E^s, E^t are Hilbert spaces. In particular, $E^1 = H_0^1(\Omega)$ if $s = 1$. Now the quadratic part $\int_\Omega \nabla u \nabla v dx$ can be well-defined [31,29,33] on E. One sees that the motivation to introduce these spaces was to extend the term $\int_\Omega \nabla u \nabla v dx$ to functions of u and v with different regularity properties. Finally, one needs to choose appropriate $t, s > 0$ such that the term $\int_\Omega G(x; u, v) dx$ is also well-defined and of class C^1.

Proposition V.8 *[31, Theorem 1.1.] Assume $\hat{p}, \hat{q} \geq 1$ and Ω is a bounded subset of $\mathbb{R}^N$ with smooth boundary. For any $t > 0, s > 0$ with $s + t = 2$, if*

(F5) $$\frac{1}{\hat{p}} \geq \frac{1}{2} - \frac{s}{N},\ \frac{1}{\hat{q}} \geq \frac{1}{2} - \frac{t}{N},$$

then the embeddings

$$E^s \to L^{\hat{p}}(\Omega),\ E^t \to L^{\hat{q}}(\Omega)$$

are continuous (resp. compact, if the inequalities in (F5) *are strict).*

Proposition V.9 *[31, Prop. 1.2.] Let $g(u,v) = \int_\Omega G(x;u,v)dx$. Under conditions (F1-F5), $g(u,v)$ is of class C^1 and its derivative $g' : E \to E$ is a compact operator.*

By Prop. V.9 and previous discussion we can conclude that the functional J in (5.49) is a C^1-functional on $E = E^s \times E^t$ with $t, s > 0$ satisfying (F5).

In view of (F5), when the space dimension $N = 2$, we can choose $s = t = 1$ so that the functional J in (5.49) is a C^1-functional on $E = E^1 \times E^1 = H_0^1(\Omega) \times H_0^1(\Omega)$. This implies that the two components u and v in (5.48) belong to the same space $H_0^1(\Omega)$. The following proposition shows that Hamiltonian elliptic system (5.48) is equivalent to a noncooperative elliptic system when u and v are defined in the same space, which particularly holds true for the case $N = 2$. Due to this equivalent relation, we also call (5.48) noncooperative systems of Hamiltonian type.

Proposition V.10 *Assume that J in (5.49) is well-defined on $E = (H_0^1(\Omega))^2$. Let $U = (u+v)/\sqrt{2}, V = (u-v)/\sqrt{2}$, then $(u,v) \in E$ is a solution to (5.48) if and only if (U,V) is a solution to the following noncooperative elliptic system*

$$\begin{cases} -\Delta U = G_U(x;U,V) & x \in \Omega, \\ -\Delta V = -G_V(x;U,V) & x \in \Omega, \\ U = V = 0 & x \in \partial\Omega, \end{cases} \tag{5.50}$$

where $G(x;U,V) \equiv G(x;u,v), G_U = \frac{G_u+G_v}{\sqrt{2}}, G_V = \frac{G_u-G_v}{\sqrt{2}}$.

Proof. Multiplying both sides of (5.48) by $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ leads to (5.50). ∎

Finally, we carry out some numerical experiments for two noncooperative systems of Hamiltonian type (5.48): the Lane-Emden system and a nonlinear biharmonic problem with Navier boundary conditions. The equivalence established in Prop. V.10 allows us to use the LMMOA to solve both systems. The tolerance used is $\varepsilon = 10^{-4}$.

Example V.4 (the Lane-Emden system [10,22]) *Let* $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$. *Taking* $G(x; u, v) \equiv G(u, v) = \frac{1}{q+1}u^{q+1} + \frac{1}{p+1}v^{p+1}$ $(p, q > 0)$ *for (5.48) gives the so-called Lane-Emden system wherein* $G_u(u, v) = u^q, G_v(u, v) = v^p$. *With Props. V.8-V.9, one can check that its associated functional* J *in (5.49) is of class* $C^1(H_0^1(\Omega) \times H_0^1(\Omega), \mathbb{R})$. *In addition, the system is called superlinear (resp. sublinear) if* $\frac{1}{q+1} + \frac{1}{p+1} < 1$ *(resp.* > 1*). Clearly,* $pq > 1$ *implies* $\frac{1}{q+1} + \frac{1}{p+1} < 1$ *(superlinear) whereas* $pq < 1$ *implies* $\frac{1}{q+1} + \frac{1}{p+1} > 1$*(sublinear). Figs. 22-23 show the profiles (left) and contour plots (right) of a positive solution to the Lane-Emden system (5.48) for the case* $p = 0.3, q = 0.6$ *(sublinear) and the case* $p = 5, q = 1.5$ *(superlinear).*

Example V.5 (nonlinear biharmonic problem) *Let* $\Omega = \{x \in \mathbb{R}^2 : |x| < 3\}$. *Take* $G(x; u, v) \equiv G(u, v) = \frac{1}{q+1}|u|^{q+1} + \frac{1}{p+1}|v|^{p+1}$ $(p, q > 0)$*, then* $G_u(u, v) = |u|^{q-1}u, G_v(u, v) = |v|^{p-1}v$. *Further, if letting* $q \equiv 1$*, then system (5.48) is equivalent to the following biharmonic problem with so-called Navier boundary conditions*

$$\begin{cases} \Delta^2 v = |v|^{p-1}v & x \in \Omega, \\ v = \Delta v = 0 & x \in \partial\Omega. \end{cases} \tag{5.51}$$

Again, one can check that its associated functional J *is of class* $C^1(H_0^1(\Omega) \times H_0^1(\Omega), \mathbb{R})$. *Figs. 24-25 depict the profiles (left) and contour plots (right) of a positive solution and a sign-changing solution to the biharmonic problem (5.51) with* $p = 3$. *In both figures,* $u = -\Delta v$. *We note that, based on our computations, the sign-changing solution to (5.51) is not observed if we reduce the radius of the domain* Ω*, e.g., when radius* $= 2$.

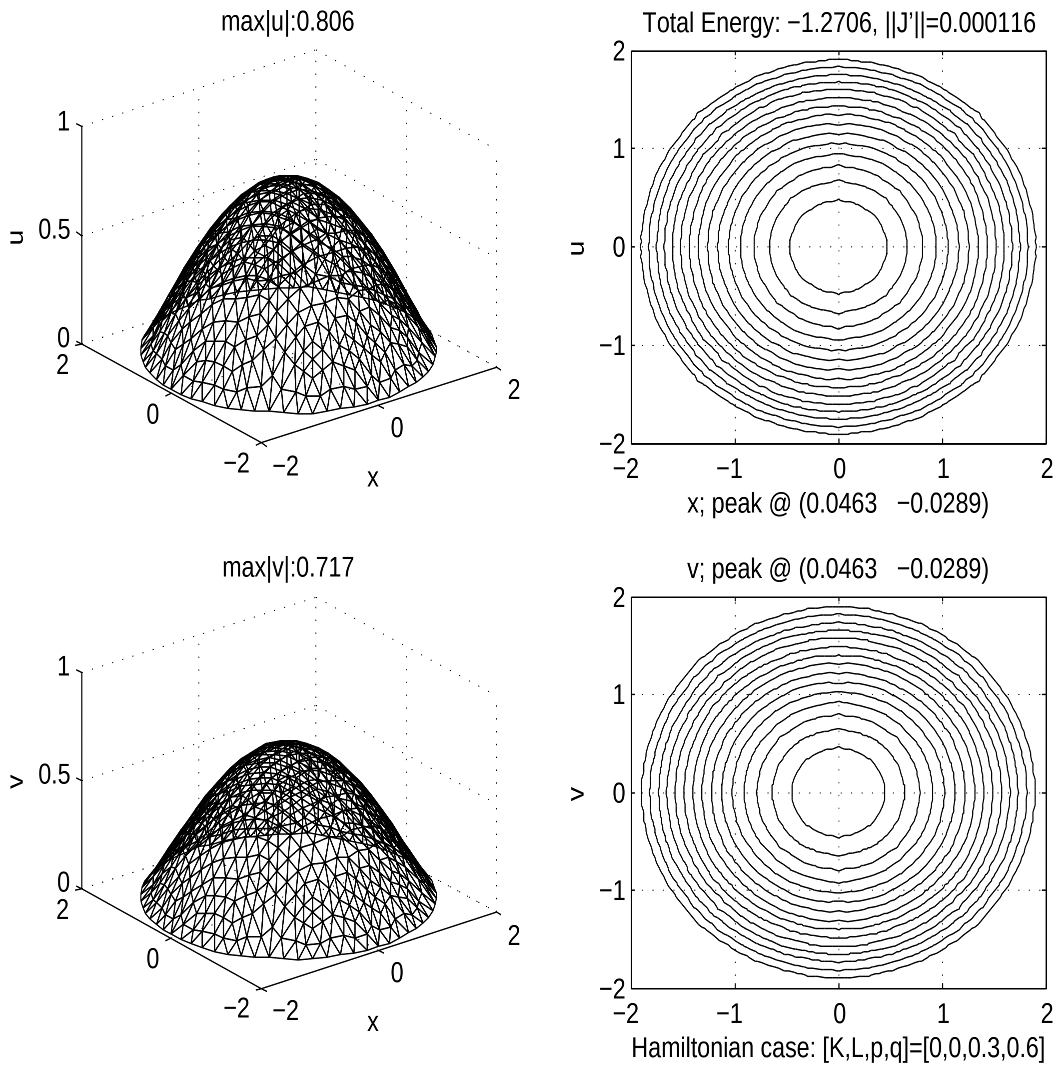

Fig. 22. A radial positive solution to the Lane-Emden system (see Example V.4) with $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$, $p = 0.3, q = 0.6:$ sublinear case.

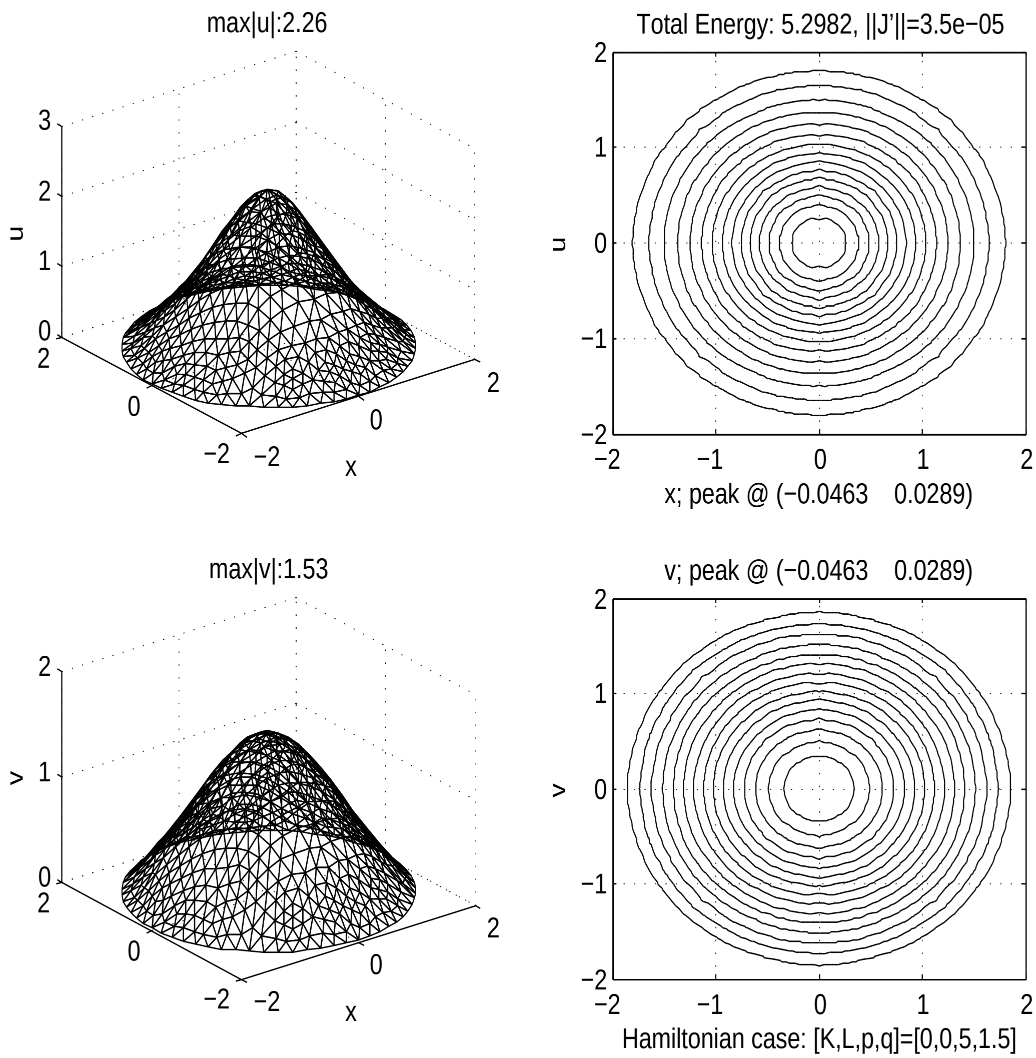

Fig. 23. A radial positive solution to the Lane-Emden system (see Example V.4) with $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$, $p = 5, q = 1.5$: superlinear case.

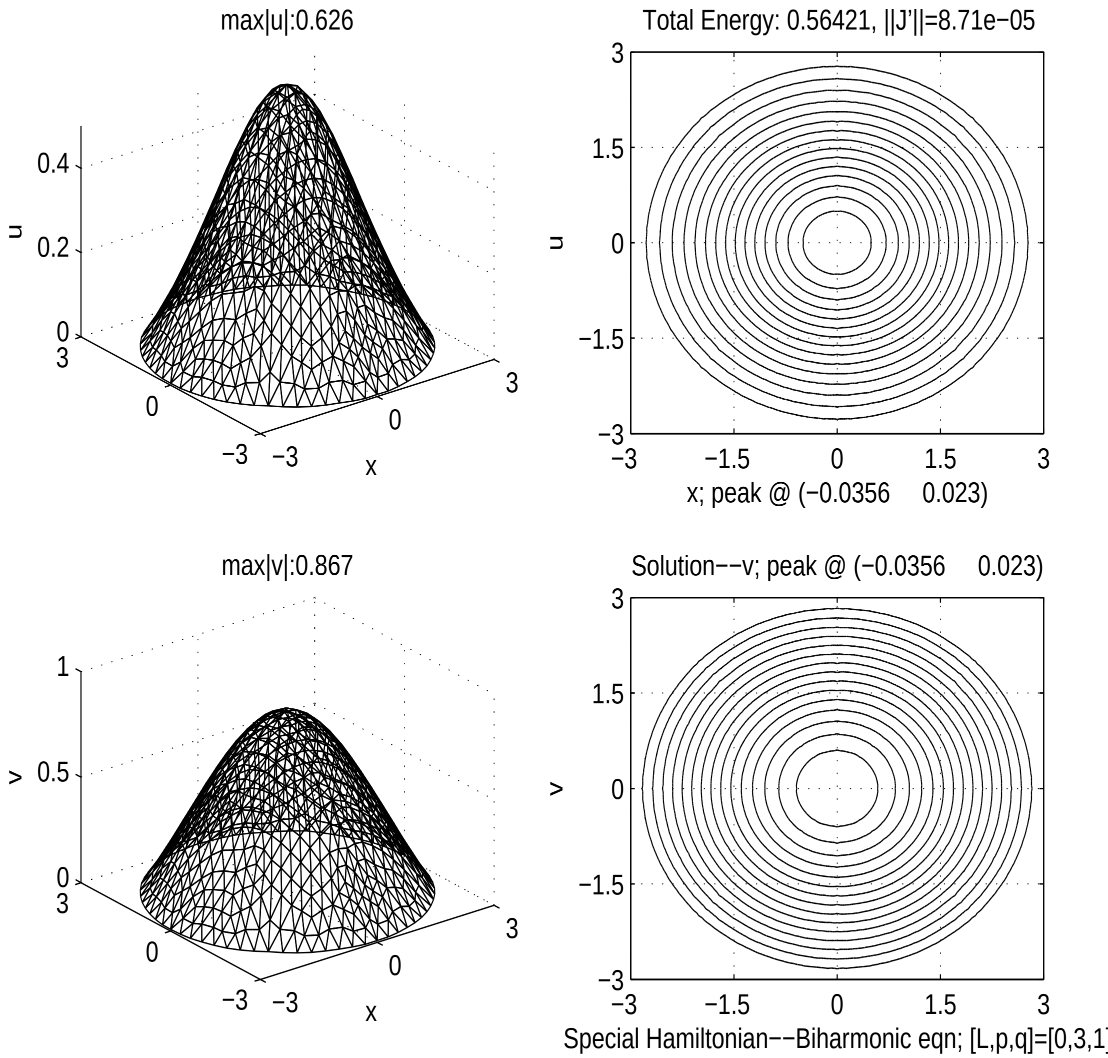

Fig. 24. A radial positive solution to the biharmonic problem (5.51) with $p = 3$, $\Omega = \{x \in \mathbb{R}^2 : |x| < 3\}$. Here, $u = -\Delta v$.

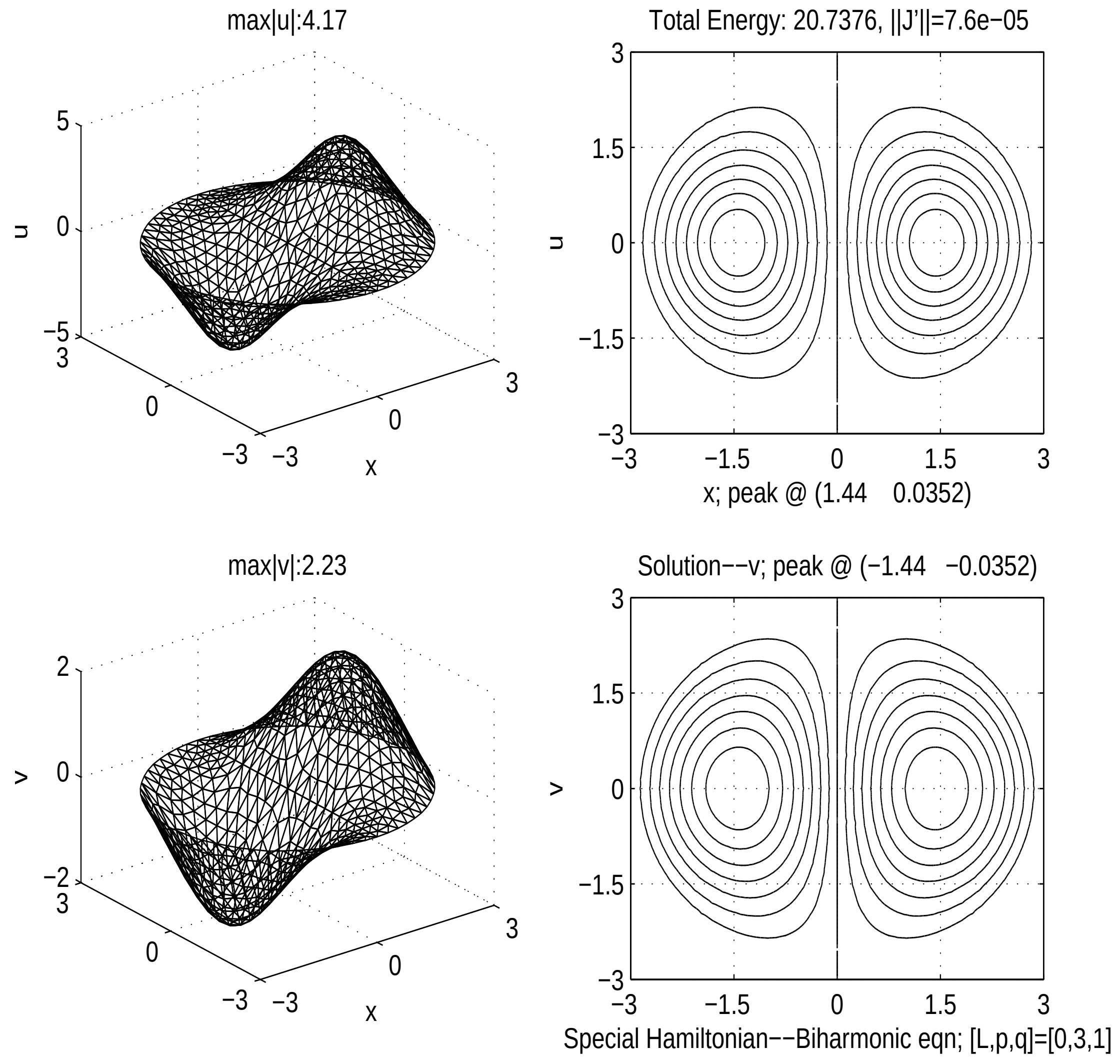

Fig. 25. A sign-changing solution to the biharmonic problem (5.51) with $p = 3$, $\Omega = \{x \in \mathbb{R}^2 : |x| < 3\}$. Here, $u = -\Delta v$.

CHAPTER VI

CONCLUSIONS

We have respectively established a local characterization for saddle points of finite Morse index and of infinite Morse index and presented two numerical methods (i.e., a local min-orthogonal method and a local min-max-orthogonal method) for three types of variational elliptic systems: cooperative, noncooperative, and Hamiltonian. Solutions to those systems correspond to coexisting saddle points or saddle points with infinite Morse index of certain definite or strongly indefinite functionals. Instability analysis on saddle points of finite Morse index is carried out via the local min-orthogonal method developed. In particular, estimates on the Morse index of coexisting saddle points are given and can be used as a guidance in numerical computations.

There are two key ingredients in developing both methods. The first one is to propose a selection function so that a solution manifold can be properly defined and used as a natural constaint which can significantly narrow our search for saddle points, especially for saddle points of infinite Morse index. Restricting our search on the solution manifold makes it easy to locate the unstable solutions, namely, the saddle points. The second one is to establish a proper characterization for saddle points so that a numerical algorithm can be appropriately designed and feasibly implemented as well.

With the help of the (PS) condition, a subsequence convergence result is established for both methods. Numerically, both methods have been applied to solve several nonlinear elliptic problems including systems derived from certain coupled NLS equations in nonlinear optics, the Lane-Emden system arising from celestial physics and the nonlinear biharmonic problem with the Navier boundary conditions.

Although our methods are quite useful and successful in finding multiple unstable solutions to several types of elliptic systems, there are still improvements that can be made in order to solve more general elliptic systems for multiple solutions. For instance, the Hamiltonian elliptic systems (5.48) are far from being completely solved. As noted in Section V.C, the LMMOM can only be applied to Hamiltonian elliptic systems which can be transformed into noncooperative systems, a very special case that occurs only when the two independent variables u and v have the same regularity or namely belong to the same space. Therefore, for general Hamiltonian elliptic systems, we need to establish a more general local characterization on unstable solutions as well as a new stable method so that their multiple solutions can be found. To do that, we may extend the L-$\perp$ selection in Definition II.1 as follows.

Let $H \equiv H_1 \times H_2$ be a product Hilbert space, L be its finite-dimensional closed subspace. Assume $L = L_1 \oplus L_2, H = H^+ \oplus H^-$ such that $H^+ = L_1 \oplus L_1^\perp, H^- = L_2 \oplus L_2^\perp$.

Definition VI.1 *Assume $J \in C^1(H, \mathbb{R})$. A set-valued mapping P: $S_{L_1^\perp \oplus L_2^\perp} \to 2^H$ is called an L-$\perp$ mapping of J w.r.t. the splitting $H = H^+ \oplus H^-$ if*

$$P(w) := \Big\{ z \in span\{L_1, \xi, L_2, \eta\} : \partial \tilde{J}_1(z) \perp span\{L_1, \xi\}, \partial \tilde{J}_2(z) \perp span\{L_2, \eta\} \Big\},$$

$\forall w = \xi \oplus \eta \in S_{L_1^\perp \oplus L_2^\perp}$ with $\xi \in H^+, \eta \in H^-$, where $\nabla J = \partial \tilde{J}_1 \oplus \partial \tilde{J}_2, \partial \tilde{J}_1 \in H^+, \partial \tilde{J}_2 \in H^-$. A single-valued mapping $p : S_{L_1^\perp \oplus L_2^\perp} \to H$ is called an L-$\perp$ selection function of J w.r.t. the splitting $H = H^+ \oplus H^-$ if $p(w) \in P(w)$.

With this new definition, condition (3.6) used in our local min-max-orthogonal characterization on saddle points (see Theorem III.1) can be replaced by

$$J(p(\tfrac{\xi^* \oplus \eta}{\|\xi^* \oplus \eta\|})) \leq J(p(\xi^* \oplus \eta^*)) \leq J(p(\tfrac{\xi \oplus \eta^*}{\|\xi \oplus \eta^*\|}))$$
$$\forall \xi \in U, \eta \in V \ \text{ with } \ \xi \oplus \eta \in S_{L_1^\perp \oplus L_2^\perp},$$

and hence the resulting characterization can be applied to more general strongly indefinite functionals. Obviously, this general local characterization on unstable solutions is worth more attention.

On the other hand, the definition of an L-$\perp$ selection in Definition II.1 may be extended in another direction so that multiple solutions to even nonvariational problems can also be found in a stable way. Consider, for example, an abstract nonvariational problem

$$\begin{cases} f(u,v) &= 0 \\ g(u,v) &= 0, \end{cases} \tag{6.1}$$

where $f = F_u$, $g = G_v$ for some functionals $F, G \in C^1(H_1 \times H_2, \mathbb{R})$ in variables (u,v). An L-$\perp$ selection of F, G w.r.t. a support $L = L_1 \times L_2$ can be defined as

$$P(w) := \Big\{ z \in [L_1, w_1] \times [L_2, w_2] : f(z) \perp [L_1, w_1], g(z) \perp [L_2, w_2] \Big\}, \forall w = (w_1, w_2) \in S_{L^\perp}.$$

This extension as well as its implications for nonvariational problems deserves more attention and is our ongoing research.

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